

ON THE VISCOSITY SOLUTIONS TO SOME NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We consider viscosity solutions of a class of nonlinear degenerate elliptic equations on bounded domains. We prove comparison principles and a priori supremum bounds for the solutions. We also address the eigenvalue problem and, in many instances, show the existence of a first eigenvalue and a first positive eigenfunction.

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1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

In this work, we study issues related to the eigenvalue problem for some nonlinear degenerate elliptic operators. This may be considered as a follow-up of the work in [6], where we showed the existence of the first eigenvalue and a positive first eigenfunction of the infinity-Laplacian. The current work continues the effort of studying similar questions for a more general class of nonlinear elliptic, possibly degenerate, operators. See [4, 8, 3, 6, 11, 12, 13, 14].

To state our results more precisely, we introduce notations that will be used through out this work. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, $\overline{\Omega}$ its closure and $\partial\Omega$ its boundary. Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous, S^n denote the set of $n \times n$ symmetric matrices and $H(p, X)$ be continuous, for $(p, X) \in \mathbb{R}^n \times S^n$. We study properties of viscosity solutions of problems of the type

$$(1.1) \quad H(Du, D^2u) + f(x, u) = 0, \text{ in } \Omega, \text{ and } u = h \text{ on } \partial\Omega,$$

where $h \in C(\partial\Omega)$. By a solution we mean a function $u \in C(\overline{\Omega})$ that solves (1.1) in the viscosity sense.

We require that the operator H satisfy monotonicity in X , homogeneity in p and X , a kind of coercivity and, in some instances, invariance under reflections and rotations, see below. Our work mainly studies questions related to the eigenvalue problem in (1.1) although some of our work applies to a more general class of functions f .

We now discuss the precise nature of H and also state the main results of this work. Let o denote the origin in \mathbb{R}^n and a point $x \in \mathbb{R}^n$ will be occasionally written as (x_1, x_2, \dots, x_n) . By I we denote the $n \times n$ identity matrix, O will denote the $n \times n$ matrix with all entries equalling zero. Also, e will always stand for a unit vector in \mathbb{R}^n .

Through out this work we require that $H \in C(\mathbb{R}^n \times S^n, \mathbb{R})$ and $H(p, O) = 0$, $\forall p \in \mathbb{R}^n$. We now describe the conditions that H satisfies.

Condition A (Monotonicity): The operator $H(p, X)$ is continuous at $p = 0$ for any $X \in S^n$ and $H(p, O) = 0$, for any $p \in \mathbb{R}^n$. In addition, for any $X, Y \in S^n$ with $X \leq Y$,

$$(1.2) \quad H(p, X) \leq H(p, Y), \quad \forall p \in \mathbb{R}^n.$$

It is clear that if $X \geq O$ then $H(p, X) \geq 0$, for any p .

Condition B (Homogeneity): There are constants $k_1 \geq 0$ and $k_2 > 0$, an odd integer, such that for any $(p, X) \in \mathbb{R}^n \times S(n)$,

$$(1.3) \quad H(\theta p, X) = |\theta|^{k_1} H(p, X), \quad \forall \theta \in \mathbb{R}, \quad \text{and} \quad H(p, \theta X) = \theta^{k_2} H(p, X), \quad \forall \theta > 0.$$

Although our work allows $k_2 \geq 1$, our interest is in the case $k_1 = 1$. Define

$$(1.4) \quad k = k_1 + k_2 \quad \text{and} \quad \gamma = k_1 + 2k_2.$$

For the next condition, we work with the matrix $e \otimes e$, where $e \in \mathbb{R}^n$ is a unit vector. Observe that $(e \otimes e)_{ij} = e_i e_j$ and $e \otimes e$ is a non-negative definite matrix.

Condition C (Coercivity): H is coercive in the following sense. Let e denote a unit vector in \mathbb{R}^n . For every $-\infty < s < \infty$, we set

$$(1.5) \quad \begin{aligned} m_1(s) &= \min_{|e|=1} H(e, I - se \otimes e), \quad m_2(s) = \max_{|e|=1} H(e, I - se \otimes e), \\ m_3(s) &= \min_{|e|=1} H(e, se \otimes e - I) \quad \text{and} \quad m_4(s) = \max_{|e|=1} H(e, se \otimes e - I). \end{aligned}$$

If H is odd in X then $m_3(s) = -m_2(s)$ and $m_4(s) = -m_1(s)$. Also, the functions $m_1(s)$ and $m_2(s)$ are decreasing in s while $m_3(s)$ and $m_4(s)$ are increasing in s . If $s < 1$ then $I - se \otimes e$ is a positive definite matrix and, by Condition A, $m_1(s) \geq 0$ and $m_4(s) \leq 0$.

Let k_1 , k_2 and $k = k_1 + k_2$ be as in (1.3) and (1.4). Set $\hat{s} = k_1/k$. We impose that

$$(1.6) \quad m_1(\hat{s}) > 0 \quad \text{and} \quad m_4(\hat{s}) < 0.$$

More generally, we require that there are $-\infty < s_1 \leq 1 \leq s_0 < \infty$ (see (1.2)) such that

$$(1.7) \quad \begin{aligned} \text{(i)} \quad & \min\{m_1(s), -m_4(s)\} > 0, \quad \forall s \leq s_1, \quad \text{and} \\ \text{(ii)} \quad & \max\{m_2(s), -m_3(s)\} < -\ell, \quad \forall s \geq s_0, \end{aligned}$$

where $0 < \ell < \infty$.

With (1.5), (1.6) and (1.7) in view, we set

$$(1.8) \quad m_1(s) = \min\{m_1(s), -m_4(s)\} \quad \text{and} \quad m_2(s) = \max\{m_2(s), -m_3(s)\}.$$

Also, with (1.4) and (1.6) in mind, we set

$$(1.9) \quad m_1 = m_1(\hat{s}), \quad m_2 = m_2(\hat{s}), \quad \alpha = \frac{k_1 + 2k_2}{k} = \frac{\gamma}{k} \quad \text{and} \quad \sigma = \frac{1}{\alpha m_1^{1/k}}.$$

In Part II of work, we will distinguish between the following two cases.

(1.10)(i) $\exists 1 < \bar{s} < 2$ such that $m_2(\bar{s}) < 0$, or (ii) $\exists \bar{s} \geq 2$ such that $m_2(s) < 0$, $\forall s > \bar{s}$.

Condition D (Symmetry): H is invariant under rotations and reflections. As a result if $v(x) = v(r)$, where $r = |x - z|$ for some $z \in \mathbb{R}^n$, then

$$(1.11) \quad H(Dv, D^2v) = G(r, v'(r), v''(r)).$$

Restated, $H(e, I - se \otimes e)$ is independent of e .

In Section 3, we discuss some examples and also make some additional comments.

We now address the eigenvalue problem. We take $a \in C(\Omega) \cap L^\infty(\Omega)$, $\inf_\Omega a > 0$. Consider the problem of finding (λ, u) where $\lambda \in \mathbb{R}$ and $u \in C(\overline{\Omega})$ solve

$$(1.12) \quad H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \text{ in } \Omega, \text{ and } u = h \text{ on } \partial\Omega,$$

where $h \in C(\partial\Omega)$ and $h > 0$. If (λ, u) solves (1.12) with $h = 0$ then we say λ is an eigenvalue of the operator H and $u \not\equiv 0$ an eigenfunction corresponding to λ . Our main effort is to characterize first eigenvalue and the first eigenfunction, see [4, 6, 11]. To this end, we study (1.12) and show the existence of positive solutions when $h > 0$ and when λ is less than a certain value $\lambda_\Omega > 0$ which turns out to be the first eigenvalue of H .

It turns out that when (1.10)(i) holds, conditions A , B and C suffice. However, (1.10)(ii) appears to be less tractable and we impose additional conditions. At this time it is not clear to us as to how to prove a Harnack's inequality for non-negative super-solutions for such a general class of operators.

We now discuss the work in [8] that addresses the eigenvalue problem for nonlinear elliptic equations. Besides homogeneity they require that the operator H satisfy $\forall (p, Y) \in \mathbb{R}^n \times S^n$,

$$a|p|^q \text{Trace}(X) \leq H(p, Y + X) - H(p, Y) \leq b|p|^q \text{Trace}(X), \quad \forall X \in S^n, X \geq 0,$$

where $0 < a \leq b < \infty$ and $q > -1$. Clearly, this condition implies that

$$(1.13) \quad \begin{aligned} (i) \quad & a(t - s) \leq H(e, I + te \otimes e) - H(e, I + se \otimes e) \leq b(t - s), \quad t \geq s, \text{ and} \\ (ii) \quad & a \leq \frac{H(e, I - e \otimes e)}{n - 1} \leq b. \end{aligned}$$

Our conditions require that $H(p, X + Y) \geq H(p, X)$, for $Y \geq 0$, and coercivity as stated in condition C. Thus, $H(e, I - se \otimes e)$ is continuous and non-increasing in s (see condition A), and that (1.6) and (1.7) hold. The last two conditions are also satisfied by the operators in [8]. However, we do not require that H be Lipschitz continuous, see (1.13)(i). Also, unlike (1.13)(ii), we allow the possibility that $H(e, I - e \otimes e) = 0$ (as in the case of the infinity-Laplacian). While [8] does not require condition D, the bounds in terms of the Laplacian support radial solutions.

These being unavailable, we require that H admit radial solutions if (1.10)(ii) holds. Also, we require that $q \geq 0$ while the work in [8] allows $q > -1$.

We also remark that our approach is different from [8]. We work the equation $H(Du, D^2u) + \lambda a(x)u^k = 0$ with positive boundary data, while in [8], the authors work with the non-homogeneous equation $H(Du, D^2u) + \lambda a(x)u^k = f(x)$ with zero boundary data. It is not clear if a version of Theorem 1.3 and some of the estimates in Section 7 hold in their case.

We now state the main results of this work. The set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, will always stand for a bounded domain in this work. By $usc(\Omega)$ we denote the class of all upper semi-continuous functions on Ω , and $lsc(\Omega)$ will denote the class of all lower semi-continuous functions on Ω .

The first result is a quotient type comparison principle for positive solutions, also see [8]. Let $g, h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $a : \Omega \rightarrow \mathbb{R}$, $a > 0$, be continuous. Suppose that $m > 0$ is such that

$$(1.14) \quad h(x, t) \geq a(x)|t|^{m-1}t > g(x, t), \quad \forall (x, t) \in \Omega \times (0, \infty)$$

Theorem 1.1. *Let H satisfy conditions A and B, and g and h be as in (1.14). Suppose that $u \in usc(\Omega)$, and $v \in lsc(\Omega)$, $v > 0$, solve*

$$H(Du, D^2u) + g(x, u) \geq 0, \quad \text{and} \quad H(Dv, D^2v) + h(x, v) \leq 0, \quad \text{in } \Omega.$$

Recall k from (1.4). Then either $u \leq 0$ in Ω , or the following conclusions hold.

(a) Suppose that $k = m$. (i) If $U \subset \Omega$ is a compactly contained sub-domain of Ω such that $u > 0$ somewhere in U then

$$\sup_U \frac{u}{v} = \sup_{\partial U} \frac{u}{v} > 0.$$

(ii) Assume that $u > 0$ somewhere in Ω , and $U_j \subset U_{j+1} \subset \Omega$, $j = 1, 2, \dots$, are compactly contained sub domains of Ω , with $\cup_j U_j = \Omega$. If $\lim_{j \rightarrow \infty} \sup_{U_j} u/v < \infty$, then

$$0 < \sup_{\Omega} \frac{u}{v} = \lim_{j \rightarrow \infty} \left(\sup_{U_j} \frac{u}{v} \right).$$

(b) Take $k \neq m$. We assume further that either (i) $k > m$ and $(u/v)(z) > 1$, for some z in Ω , or (ii) $k < m$ and $\sup_{\Omega} u/v < 1$. Then the conclusions in (i) and (ii) of part (a) hold.

A related version of the comparison principle for a somewhat general case is discussed in Section 5. See [8].

For the remaining results, we assume that H satisfies conditions A, B and C.

We now state a result on a priori supremum bounds that is useful for the eigenvalue problem for H on Ω . Let $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and

$$(1.15) \quad \sup_{\Omega \times [t_1, t_2]} |f(x, t)| < \infty, \quad \forall t_1, t_2 \text{ such that } -\infty < t_1 \leq t_2 < \infty.$$

Assume that there are constants $-\infty < \mu_1 \leq 0 \leq \mu_2 < \infty$ such that

$$(1.16) \quad \limsup_{t \rightarrow \infty} \left(\frac{\sup_{\Omega} f(x, t)}{t^k} \right) \leq \mu_2 \quad \text{and} \quad \liminf_{t \rightarrow -\infty} \left(\frac{\inf_{\Omega} f(x, t)}{|t|^k} \right) \geq \mu_1.$$

Theorem 1.2. *Let $\lambda \in \mathbb{R}$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be as in (1.15) and (1.16), and $h \in C(\partial\Omega)$. Suppose that $u \in C(\overline{\Omega})$ solves*

$$H(Du, D^2u) + \lambda f(x, u) = 0, \quad \text{in } \Omega, \text{ and } u = h \text{ on } \partial\Omega.$$

(a) *If $|\lambda|$ is small enough then u is a priori bounded and $\sup_{\Omega} |u| \leq K$, where K depends on $\lambda, \mu_1, \mu_2, k, h$ and Ω .*

(b) *If $\mu_1 = \mu_2 = 0$ then, for any λ , u is a priori bounded and $\sup_{\Omega} |u| \leq K$, where K depends on λ, k, h and Ω .*

Let $\delta > 0$ and $\lambda > 0$. Consider the problem of finding a positive solution $u_{\lambda} \in C(\overline{\Omega})$ of

$$(1.17) \quad H(Du_{\lambda}, D^2u_{\lambda}) + \lambda a(x)u_{\lambda}^k = 0, \quad \text{in } \Omega, \text{ and } u_{\lambda} = \delta \text{ on } \partial\Omega.$$

We call

$$(1.18) \quad \lambda_{\Omega} = \sup\{\lambda : (1.17) \text{ has a positive solution } u_{\lambda}\}.$$

We show in Theorem 1.4 that $\lambda_{\Omega} > 0$.

The next result shows that a positive solution u_{λ} , for any $0 < \lambda < \lambda_{\Omega}$, is an increasing Lipschitz continuous function of λ .

Theorem 1.3. *Let $\lambda > 0$, $\delta > 0$ and $u_{\lambda} \in C(\overline{\Omega})$, $u_{\lambda} > 0$, solve*

$$(1.19) \quad H(Du_{\lambda}, D^2u_{\lambda}) + \lambda a(x)u_{\lambda}^k = 0, \quad \text{in } \Omega \text{ and } u_{\lambda} = \delta \text{ on } \partial\Omega.$$

Set $v_x(\lambda) = u_{\lambda}(x)$, $\forall x \in \overline{\Omega}$ and $M_{\lambda} = \sup_{\Omega} u_{\lambda}$. Then for each $x \in \Omega$, $v_x(\lambda)$ is a non-decreasing Lipschitz continuous function of λ and for a.e. λ ,

$$\frac{v_x(\lambda) \log(v_x(\lambda)/\delta)}{k\lambda} \leq \frac{dv_x(\lambda)}{d\lambda} \leq \left(\frac{M_{\lambda}}{k\delta} \right) \frac{v_x(\lambda) - \delta}{\lambda}, \quad 0 < \lambda < \lambda_{\Omega}.$$

Also, see Remark 7.5).

We now provide existence results for the eigenvalue problem in (1.12). The conditions (1.10) (i) and (ii) play a crucial role in these statements and the following will be assumed through out this work.

Ω is any bounded domain if (1.10)(i) holds, and Ω satisfies a uniform outer ball condition if (1.10) (ii) holds.

Theorem 1.4. *Suppose that $\lambda > 0$, and $a(x) \in C(\Omega)$ with $\inf_{\Omega} a > 0$. For $h \in C(\partial\Omega)$ with $\inf_{\partial\Omega} h > 0$, consider the boundary value problem*

$$(1.21) \quad H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega \text{ and } u = h \text{ on } \partial\Omega.$$

Set $R = \text{diam}(\Omega)$ and $\nu = \sup_{\Omega} a(x)$. Recall (1.20).

(a) If (1.10)(i) holds and $0 < \lambda < |m_2(\bar{s})|(2 - \bar{s})^k(\nu R^\gamma)^{-1}$ then (1.21) has a unique positive solution.

(b) Suppose that (1.10)(ii) holds and Ω satisfies a uniform outer ball condition with optimal radius $2\rho > 0$. Fix $\beta > \bar{s} - 2$ and $s = \beta + 2$. If

$$0 < \lambda < \frac{|m_2(s)|\beta^k}{\nu R^\gamma} \left(\frac{\rho}{R}\right)^{k\beta}$$

then (1.21) has a unique positive solution. Moreover, $u > \inf_{\partial\Omega} h$ in Ω .

We show next that λ_Ω , as defined in (1.18), is independent of h . Also, see Remark 8.3.

Theorem 1.5. Suppose that (1.20) holds. Let $\delta > 0$, $a(x) \in C(\Omega)$, $\inf_{\Omega} a > 0$, and $h \in C(\partial\Omega)$, with $\inf_{\partial\Omega} h > 0$. Suppose that, for some $\lambda > 0$, the problem $H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0$, in Ω , and $u = \delta$ on $\partial\Omega$, has a positive solution. Then the problem

$$(1.22) \quad H(Dv, D^2v) + \lambda a(x)|v|^{k-1}v = 0, \quad \text{in } \Omega \text{ and } v = h \text{ on } \partial\Omega,$$

also has a positive solution.

The boundedness of λ_Ω is shown in

Theorem 1.6. Suppose that H satisfies conditions A, B and C. Let $\delta > 0$ and $a(x) \in C(\Omega) \cap L^\infty(\Omega)$, $\inf_{\Omega} a > 0$. Recall (1.20).

(a) Suppose that (1.10) (i) holds then $\lambda_\Omega < \infty$. (b) Suppose that (1.10) (ii) holds and H satisfies D then $\lambda_\Omega < \infty$.

In part (ii), if a Harnack's inequality holds then the conclusion follows without the imposition of condition D. See Remark 9.2.

Finally, we show

Theorem 1.7. Suppose that H satisfies conditions A, B and C. Let $a \in C(\Omega, \mathbb{R})$, $\inf_{\Omega} a > 0$. Consider the problem

$$(1.23) \quad H(Du, D^2u) + \lambda_\Omega a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega,$$

where $u \in C(\overline{\Omega})$. Recall (1.20).

(i) Suppose that (1.10) (i) holds then (1.23) has a positive eigenfunction u .

(ii) Suppose that H satisfies D and (1.10) (ii) holds then (1.23) has a positive radial eigenfunction when Ω is a ball.

At this time, it is not clear to us as to how to extend part (ii) to general domains. Also, our work does not address whether λ_Ω is simple or isolated.

We describe the lay out of the work. In Section 2, we include definitions, notations and some useful calculations. Section 3 contains examples and some further discussion. Section 4 presents

comparison principles when H satisfies condition A and are of some what general nature. The remaining work is divided into two parts. Part I has Sections 5 and 6. Sections 7-10 are in Part II. Section 5 lists additional comparison principles under the conditions A and B and the proof of Theorem 1.1. We also include a change of variables formula, important for Theorem 1.6. Sections 6-10 we assume that H satisfies A , B and C . The proofs of Theorems 6.4 and 1.2 are in Section 6. Section 7 contains a discussion of questions related to the problem (1.17) and shows that solutions u_λ are Lipschitz continuous in λ . Proofs of Theorems 1.4 and 1.5 are in Section 8. Theorem 1.6 is proven in Section 9. We present a proof of Theorem 1.7 in Section 10.

2. ADDITIONAL NOTATIONS, DEFINITIONS AND CALCULATIONS

We introduce additional notations and definitions. We use o to denote the origin. By $B_s(p)$, $s > 0$, we mean the ball of radius s centered at p . In this work, all differential equations and inequalities will be understood in the sense of viscosity, see below and [9]. We assume through out that $H \in C(\mathbb{R}^n \times S^n, \mathbb{R})$ and satisfies condition A , see (1.2).

We define the notion of a viscosity solution u to the following in Ω ,

$$(2.1) \quad H(Du, D^2u) + f(x, u) = 0 \quad \text{in } \Omega \text{ and } u = h \text{ on } \partial\Omega,$$

where $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and $h \in C(\partial\Omega)$.

A function $u \in usc(\Omega)$ is said to be a viscosity sub-solution of the equation in (2.1), in Ω , or solves $H(Du, D^2u) + f(x, u) \geq 0$, in Ω , if the following holds. For any $\psi \in C^2(\Omega)$ such that $u - \psi$ has a maximum at a point $y \in \Omega$, we have

$$H(D\psi(y), D^2\psi(y)) + f(y, u(y)) \geq 0.$$

Similarly, $u \in lsc(\Omega)$ is said to be a viscosity super-solution of the equation (2.1) or solves $H(Du, D^2u) + f(x, u) \leq 0$, in Ω , if, for any $\psi \in C^2(\Omega)$ such that $u - \psi$ has a minimum at $y \in \Omega$, we have

$$H(D\psi(y), D^2\psi(y)) + f(y, u(y)) \leq 0.$$

A function $u \in C(\Omega)$ is a viscosity solution of if it is both a sub-solution and a super-solution.

We define $u \in usc(\overline{\Omega})$ to be a viscosity sub-solution to the problem (2.1) if u is a sub-solution in Ω and $u \leq h$ on $\partial\Omega$. Similarly, $u \in lsc(\overline{\Omega})$ is a super-solution of (2.1) if u is a super-solution in Ω and $u \geq h$ on $\partial\Omega$. We define $u \in C(\overline{\Omega})$ to be a solution to (2.1), if it is both a sub-solution and a super-solution of (2.1).

In this work, we will utilize radial sub-solutions and super-solutions. We discuss (1.10) in this context. Let $v(x) = v(r)$ where $r = |x - z|$, for some $z \in \mathbb{R}^n$. Set $e = (e_1, e_2, \dots, e_n)$ where $e_i = (x - z)_i / r$, $i = 1, 2, \dots, n$. Then for $x \neq z$,

$$(2.2) \quad H(Dv, D^2v) = H\left(v'(r)e, \frac{v'(r)}{r}(I - e \otimes e) + v''(r)e \otimes e\right),$$

where I is the $n \times n$ identity matrix. We now impose conditions A , B and C on H . Recall (1.3)-(1.10),

$$k = k_1 + k_2, \quad \gamma = k_1 + 2k_2 \quad \text{and} \quad \alpha = \gamma/k.$$

Then (2.2) reads

$$(2.3) \quad H(Dv, D^2v) = \frac{|v'(r)|^k}{r^{k_2}} H \left[e, \pm \left\{ I - \left(1 - \frac{rv''(r)}{v'(r)} \right) e \otimes e \right\} \right].$$

Take $v(r) = c \pm dr^\beta$, where $d > 0$ and $\beta > 0$. Using (2.3). We obtain, in $r > 0$,

$$(2.4) \quad \begin{aligned} H(Dv, D^2v) &= H \left[\pm d\beta r^{\beta-1} e, \pm d\beta r^{\beta-2} \{ I - (2-\beta)e \otimes e \} \right] \\ &= d^k \beta^k r^{(\beta-1)k_1 + (\beta-2)k_2} H [e, \pm \{ I - (2-\beta)e \otimes e \}] \\ &= (d\beta)^k r^{k\beta-\gamma} H [e, \pm \{ I - (2-\beta)e \otimes e \}]. \end{aligned}$$

If $v = c \pm dr^{-\beta}$, where $d > 0$ and $\beta > 0$, then (2.4) yields

$$(2.5) \quad H(Dv, D^2v) = \frac{(d\beta)^k}{r^{k\beta+\gamma}} H [e, \mp \{ I - (\beta+2)e \otimes e \}].$$

If (1.10)(i) holds and $v^\pm = c \pm dr^\beta$ with $\beta = 2 - \bar{s} > 0$ ($0 < \beta < 1$) then (1.8) and (2.4) imply

$$(2.6) \quad \begin{aligned} H(Dv^+, D^2v^+) &= (d\beta)^k r^{k\beta-\gamma} H(e, I - \bar{s}e \otimes e) \leq -\frac{(d\beta)^k |m_2(\bar{s})|}{r^{\gamma-k\beta}} < 0, \quad \text{and} \\ H(Dv^-, D^2v^-) &= (d\beta)^k r^{k\beta-\gamma} H(e, \bar{s}e \otimes e - I) \geq \frac{(d\beta)^k |m_2(\bar{s})|}{r^{\gamma-k\beta}} > 0. \end{aligned}$$

If (1.10)(ii) holds and $v^\pm = c \pm dr^{-\beta}$ with $\beta > \bar{s} - 2 > 0$, then (1.8) and (2.5) lead to

$$(2.7) \quad \begin{aligned} H(Dv^+, D^2v^+) &= \frac{(d\beta)^k}{r^{k\beta+\gamma}} H(e, \bar{s}e \otimes e - I) \geq \frac{(d\beta)^k |m_2(s)|}{r^{k\beta+\gamma}} > 0, \quad \text{and} \\ H(Dv^-, D^2v^-) &= \frac{(d\beta)^k}{r^{k\beta+\gamma}} H(e, I - \bar{s}e \otimes e) \leq -\frac{(d\beta)^k |m_2(s)|}{r^{k\beta+\gamma}} < 0, \end{aligned}$$

where $s = \beta + 2$. As a second application, set $\beta = \alpha = \gamma/k$ in (2.4) (see (1.9)) and take $v^\pm = c \pm dr^\alpha$, $d > 0$, to obtain

$$(2.8) \quad \begin{aligned} H(Dv^+, D^2v^+) &= (d\alpha)^k H \left(e, I - \frac{k_1}{k} e \otimes e \right) \geq (d\alpha)^k m_1 = \left(\frac{d}{\sigma} \right)^k > 0, \quad \text{and} \\ H(Dv^-, D^2v^-) &= (d\alpha)^k H \left(e, \frac{k_1}{k} e \otimes e - I \right) \leq -(d\alpha)^k m_1 = -\left(\frac{d}{\sigma} \right)^k < 0. \end{aligned}$$

Remark 2.1. Our results on existence use Perron's method, see [9]. The idea is as follows. Consider the problem of showing the existence of a solution $u \in C(\bar{\Omega})$ of

$$H(Du, D^2u) + f(x, u) = 0, \quad \text{in } \Omega \text{ and } u = h \text{ on } \partial\Omega,$$

where $h \in C(\partial\Omega)$. Assume that the above admits a comparison principle. Let $\varepsilon > 0$, be a given small number. For each $y \in \partial\Omega$, we construct (i) a sub-solution v such that $v(y) = h(y) - \varepsilon$ and $v \leq h$ on $\partial\Omega$, and (ii) a super-solution w such that $w(y) = h(y) + \varepsilon$ and $w \geq h$ on $\partial\Omega$. This implies existence of a solution u . \square

3. EXAMPLES AND FURTHER COMMENTS

We discuss some examples to which our results apply. Let $s \in \mathbb{R}$ and the vector $e \in \mathbb{R}^n$ be such that $|e| = 1$. Set $r = |x|$, $\forall x \in \mathbb{R}^n$.

(i) p -Laplacian and the pseudo p -Laplacian: The p -Laplacian Δ_p , $p \geq 2$, can be written as $D_p u = |Du|^{p-2} \Delta u + (p-2)|Du|^{p-4} \Delta_\infty u$, where $\Delta_\infty u = \sum_{i,j=1}^n D_i u D_j u D_{ij} u$ is the infinity-Laplacian. It is easy to see that $H(e, I - se \otimes e) = (n + p - 2) - (p - 1)s$.

We consider a more general version i.e., $H(Du, D^2 u) = |Du|^q \Delta u + a|Du|^{q-2} \Delta_\infty u$, $q \geq 0$, then $H(e, I - se \otimes e) = n - s + a(1 - s)$, $\forall s$. Thus,

$$\min(1, 1 + a) \leq \frac{H(e, I - se \otimes e)}{n - s} \leq \max(1, 1 + a), \quad \forall a > -1 \text{ and } \forall s \leq 1.$$

If $a > -1$ then conditions A, B, C and D, in Section 1, are satisfied. Hence, all the results of this work hold.

Next we discuss a version of the pseudo p -Laplacian, which we denote by Δ_p^s , where

$$\Delta_p^s u = |Du|^q \sum_{i=1}^n |D_i u|^p D_{ii} u, \quad p, q \geq 0, \quad \text{and } H = \Delta_p^s.$$

We observe, using Holder's inequality and that $|e_i| \leq 1$, that $\sum_{i=1}^n |e_i|^{p+2} \leq \sum_{i=1}^n |e_i|^p$,

$$\min\left(1, n^{(2-p)/2}\right) \leq \sum_{i=1}^n |e_i|^p \leq n, \quad \text{and} \quad \sum_{i=1}^n |e_i|^p \leq \left(\sum_{i=1}^n |e_i|^{p+2}\right)^{p/(p+2)} n^{2/(p+2)}.$$

Clearly, $H(e, I - se \otimes e) = \sum_{i=1}^n |e_i|^p - s \sum_{i=1}^n |e_i|^{p+2}$. Using the above, if $s > 0$ then

$$\begin{aligned} \frac{1-s}{n^{(2-p)/2}} \leq H(e, I - se \otimes e) &\leq \left(\sum_{i=1}^n |e_i|^{p+2}\right)^{p/(p+2)} \left[n^{2/(p+2)} - s \left(\sum_{i=1}^n |e_i|^{p+2}\right)^{2/(p+2)} \right] \\ &\leq \left[n^{2/(p+2)} - \frac{s}{n^{p/(p+2)}} \right] = \frac{n-s}{n^{p/(p+2)}}. \end{aligned}$$

If $e_i = 1$, for some i , then $H(e, I - se \otimes e) = 1 - s$. Also, if $e_i = k^{-1/2}$, for $i = 1, 2, 3, \dots, k$, and $e_i = 0$, $i = k+1, \dots, n$, then $H(e, I - se \otimes e) = k^{-p/2}(k - s)$, $k = 1, \dots, n$.

Conditions A, B and C hold and hence, Theorems 1.1-1.5 follow. The operator does not have radial symmetry. However, a proof of the Harnack inequality and local Lipschitz regularity may be found in [3] (see Theorem 2.9 and Remark 2.11), also see [5]. Clearly, Theorem 1.6 holds, see Remark 9.2. Moreover, a version of Lemma 10.1 holds. Also, $H(Dr, D^2 r) = r^{-1} H(e, I - e \otimes e) \geq 0$, the proof in Subsection I (take $\beta = 1$ in (b)) shows that Theorem 1.7 also holds.

(ii) ∞ -Laplacian and a related operator: Set $H(Du, D^2 u) = \Delta_\infty u = \sum_{i,j=1}^n D_i u D_j u D_{ij} u$. Thus,

$$H(e, I - se \otimes e) = \sum_{i,j=1}^n e_i e_j (\delta_{ij} - se_i e_j) = \sum_{i=1}^n e_i^2 - s \left(\sum_{i=1}^n e_i^2 \right)^2 = 1 - s.$$

Clearly, all the conditions are met and all the results stated in Section 1 hold.

For the following, we consider $q \geq 0$ and define $H(Du, D^2u) = \sum_{i,j=1}^n |D_i u|^q |D_j u|^q D_i u D_j u D_{ij} u$. Then

$$H(e, I - se \otimes e) = \sum_{i=1}^n |e_i|^{2q+2} - s \left(\sum_{i=1}^n |e_i|^{q+2} \right)^2.$$

If $s \leq 0$ then $H \geq 0$. Taking $s \geq 0$, writing $q+2 = (q+1)+1$ and observing that $(\sum_{i=1}^n |e_i|^{q+2})^2 \leq \sum_{i=1}^n |e_i|^{2q+2}$, we get

$$H(e, I - se \otimes e) \geq (1-s) \sum_{i=1}^n |e_i|^{2q+2} \geq \frac{1-s}{n^q}.$$

Next, setting $\theta = (2q+2)/(q+2)$ and using $\sum_{i=1}^n |e_i|^{2q+2} \leq (\sum_{i=1}^n |e_i|^{q+2})^\theta$, we get

$$\begin{aligned} H(e, I - se \otimes e) &\leq \left(\sum_{i=1}^n |e_i|^{q+2} \right)^\theta \left(1 - s \left(\sum_{i=1}^n |e_i|^{q+2} \right)^{2/(q+2)} \right) \\ &\leq 1 - \frac{s}{n^{q/(q+2)}} \end{aligned}$$

Thus, Theorems 1.1-1.5 hold. The operator has no radial symmetry. A Harnack's inequality and local Lipschitz continuity may be worked out along the lines of Theorem 2.9 and Remark 2.11 in [3]. Clearly, Theorem 1.6 holds. Observing that $H(Dr, D^2r) = r^{-1}H(e, I - e \otimes e) \geq 0$, one can use the proof in Subsection I (take $\beta = 1$ in (b)) to show that Theorem 1.7 also holds.

(iii) Pucci operators: Let a_i denote an eigenvalue of the matrix D^2u . For $0 < \lambda \leq \Lambda$ and $q \geq 0$ define

$$M_{\lambda, \Lambda}^{+, q}(u) = |Du|^q \left(\Lambda \sum_{a_i \geq 0} a_i + \lambda \sum_{a_i \leq 0} a_i \right) \quad \text{and} \quad M_{\lambda, \Lambda}^{-, q}(u) = |Du|^q \left(\lambda \sum_{a_i \geq 0} a_i + \Lambda \sum_{a_i \leq 0} a_i \right).$$

For any e , the eigenvalues of $I - se \otimes e$ are 1, with multiplicity $n-1$, and $1-s$. Set $H^\pm(Du, D^2u) = M_{\lambda, \Lambda}^{\pm, q}(u)$ and observe that $H^+(e, \pm(I - se \otimes e)) = -H^-(e, \mp(I - se \otimes e))$. If $s \leq 1$, then

$$H^+(e, I - se \otimes e) = \Lambda(n-s) \quad \text{and} \quad H^+(e, se \otimes s - I) = -\lambda(n-s).$$

If $s > 1$, then

$$H^+(e, I - se \otimes e) = \Lambda(n-1) + \lambda(1-s) \quad \text{and} \quad H^+(e, se \otimes s - I) = -\lambda(n-1) - \Lambda(1-s).$$

The operators H^\pm are radially symmetric. The conditions A, B, C and D are satisfied and Theorems 1.1-1.5, 1.6(b) and 1.7(ii) hold.

We can also consider the maximal and minimal Pucci operators [10](Chap 17). Let

$$J = \{M(x) \in S^n : \sum_{i,j=1}^n M_{ij}(x) \eta_i \eta_j \geq a|\eta|^2 \text{ and } \sum_{i=1}^n M_{ii}(x) = 1\} \quad \text{and} \quad 0 < a \leq 1/n.$$

We set $H^{+(-)}(Du, D^2u) = \sup(\inf)_{M(x) \in J} |Du|^{k_1} \text{Trace}(M(x)D^2u)$, $x \in \Omega$. Then $H^+(p, X) = -H^-(p, -X)$. Set $E = a(n-s)$ and $F = (1-na)$. Then for any e (see [10]),

$$H^+(e, I - se \otimes e) = E + F, \quad s \geq 0, \quad \text{and} \quad H^+(e, I - se \otimes e) = E + F(1-s), \quad s \leq 0,$$

$$H^-(e, I - se \otimes e) = E + F(1-s), \quad s \geq 0, \quad \text{and} \quad H^-(e, I - se \otimes e) = E + F, \quad s \leq 0.$$

Conditions A, B, C and D are satisfied and Theorems 1.1-1.5, 1.6(b) and 1.7(ii) hold. See [14] in this context.

4. COMPARISON PRINCIPLES UNDER CONDITION A

We assume H satisfies condition A (see (1.2)) i.e., $H(p, X)$ is continuous on $\mathbb{R}^n \times S^n$, $H(p, O) = 0$ and, for any $X, Y \in S^n$ with $X \leq Y$, we have $H(p, X) \leq H(p, Y)$, $\forall p \in \mathbb{R}^n$. The section begins with a version of a comparison principle that is used often in this work, also see [6].

Theorem 4.1. (*Comparison Principle*) *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose that $u \in \text{usc}(\overline{\Omega})$ and $v \in \text{lsc}(\overline{\Omega})$ satisfy in the viscosity sense,*

$$H(Du, D^2u) + f(x, u(x)) \geq 0 \quad \text{and} \quad H(Dv, D^2v) + g(x, v(x)) \leq 0, \quad \text{in } \Omega.$$

If $\sup_{\Omega}(u - v) > \sup_{\partial\Omega}(u - v)$ then there is a point $z \in \Omega$ such that

$$(u - v)(z) = \sup_{\Omega}(u - v) \quad \text{and} \quad g(z, v(z)) \leq f(z, u(z)).$$

Equivalently, if $\inf_{\Omega}(v - u) < \inf_{\partial\Omega}(v - u)$ then there is a point $z \in \Omega$ such that $(v - u)(z) = \inf_{\Omega}(v - u)$ and $g(z, u(z)) \leq f(z, v(z))$.

Proof. We provide an outline, see [9]. We prove part (a), the proof of part (b) follows similarly. Set $M = \sup_{\Omega}(u - v)$; define, for $\varepsilon > 0$,

$$(4.1) \quad w_{\varepsilon}(x, y) := u(x) - v(y) - \frac{1}{2\varepsilon}|x - y|^2, \quad \forall (x, y) \in \Omega \times \Omega.$$

Set $M_{\varepsilon} := \sup_{\Omega \times \Omega} w_{\varepsilon}(x, y)$, and let $(x_{\varepsilon}, y_{\varepsilon}) \in \overline{\Omega} \times \overline{\Omega}$ be such that M_{ε} is attained at $(x_{\varepsilon}, y_{\varepsilon})$.

There is a $z \in \overline{\Omega}$ such that x_{ε} and $y_{\varepsilon} \rightarrow z$, as $\varepsilon \rightarrow 0$, and $M = (u - v)(z)$. Since $(x_{\varepsilon}, y_{\varepsilon})$ is a point of maximum of $w_{\varepsilon}(x, y)$, there exist X_{ε} and Y_{ε} such that $((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, X_{\varepsilon}) \in \bar{J}^{2,+}u(x_{\varepsilon})$ and $((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, Y_{\varepsilon}) \in \bar{J}^{2,-}v(y_{\varepsilon})$. Moreover, we have $X_{\varepsilon} \leq Y_{\varepsilon}$, and using the definitions of $\bar{J}^{2,+}$ and $\bar{J}^{2,-}$, we see that

$$(4.2) \quad -f(x_{\varepsilon}, u(x_{\varepsilon})) \leq H((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, X_{\varepsilon}) \leq H((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, Y_{\varepsilon}) \leq -g(y_{\varepsilon}, v(y_{\varepsilon})).$$

Now let $\varepsilon \rightarrow 0$ to conclude that $g(z, v(z)) \leq f(z, u(z))$. □

As a consequence of Theorem 4.1, we state a version of the strong maximum principle that holds under some restrictions on f and the boundary data. See [2] for a more general version.

Lemma 4.2. (*Maximum Principle*) Let $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$. Let $J = \{c \in \mathbb{R} : f(x, c) = 0, \text{ for some } x \in \Omega\}$. Call $c_{\inf} = \inf J$, $c_{\sup} = \sup J$ and assume that $-\infty < c_{\inf} \leq c_{\sup} < \infty$.

(i) Suppose that $f \leq 0$ and $u \in \text{usc}(\overline{\Omega})$ solves $H(Du, D^2u) + f(x, u) \geq 0$, in Ω .

If $\sup_{\partial\Omega} u > c_{\sup}$ or $\sup_{\Omega} u < c_{\inf}$ then

$$u(x) < \sup_{\partial\Omega} u, \quad \forall x \in \Omega, \quad \text{and} \quad \sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u.$$

(ii) Suppose that $f \geq 0$ and $u \in \text{lsc}(\overline{\Omega})$ solves $H(Du, D^2u) + f(x, u) \leq 0$, in Ω . If $\inf_{\partial\Omega} u < c_{\inf}$ or $\inf_{\Omega} u > c_{\sup}$ then

$$u(x) > \inf_{\partial\Omega} u, \quad \forall x \in \Omega, \quad \text{and} \quad \inf_{\overline{\Omega}} u = \inf_{\partial\Omega} u$$

If in part (i) $f < 0$, and in part (ii) $f > 0$ then the corresponding conclusions hold without the stated restrictions on u in $\partial\Omega$ and Ω .

Proof. We prove (i), the proof of (ii) is similar. Suppose that the claim is false i.e., there is a point $z \in \Omega$ such that $u(z) = \sup_{\Omega} u \geq \sup_{\partial\Omega} u$. By our hypothesis, $u(z) \notin J$ i.e., $f(z, u(z)) \neq 0$. For $\varepsilon > 0$, small, define $\psi_\varepsilon(x) = u(z) + \varepsilon|x - z|^2$ in Ω . Then $\psi_\varepsilon \in C^2(\Omega)$, $(u - \psi_\varepsilon)(z) = 0$ and $(u - \psi_\varepsilon)(x) \leq -\varepsilon|x - z|^2 < 0$, $\forall x \in \overline{\Omega}$, $x \neq z$. Thus, for any $\varepsilon > 0$, z is the only point of maximum of $u - \psi_\varepsilon$ in Ω . Using the definition of a viscosity sub-solution, we have

$$H(D\psi_\varepsilon(z), D^2\psi_\varepsilon(z)) + f(z, u(z)) = H(0, 2\varepsilon I) + f(z, u(z)) \geq 0.$$

Letting $\varepsilon \rightarrow 0$, we get $0 = H(0, 0) \geq -f(z, u(z)) \geq 0$. Thus, the claim holds. \square

Finally,

Lemma 4.3. Let $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose that (i) $g(x, t) < h(x, t)$, $\forall (x, t) \in \Omega \times \mathbb{R}$ and at least one of $g(\cdot, t)$ and $h(\cdot, t)$ is non-increasing in t , or (ii) $g = h$ and g is strictly decreasing in t .

Let $u \in \text{usc}(\overline{\Omega})$ and $v \in \text{lsc}(\overline{\Omega})$ satisfy $H(Du, D^2u) + g(x, u) \geq 0$ and $H(Dv, D^2v) + h(x, v) \leq 0$, in Ω . If $u \leq v$, on $\partial\Omega$, then $u \leq v$ in Ω . Moreover,

$$\sup_{\Omega} (u - v)^+ = \sup_{\partial\Omega} (u - v)^+.$$

Proof. Suppose that g is non-increasing in t and $\sup_{\Omega} (u - v) > \sup_{\partial\Omega} (u - v)$. By Theorem 4.1, there is a point $z \in \Omega$ such that $(u - v)(z) = \sup_{\Omega} (u - v) > 0$ and $g(z, u(z)) \geq h(z, v(z))$. Since $u(z) > v(z)$, we get $g(z, v(z)) \geq g(z, u(z)) \geq h(z, v(z))$, a contradiction. Thus, the claim holds.

Next, set $\mu = \sup_{\partial\Omega} (u - v)$ and assume $\mu > 0$. Define $u_\mu = u - \mu$ and observe that

$$H(Du_\mu, D^2u_\mu) = H(Du, D^2u) \geq -g(x, u) \geq -g(x, u_\mu), \quad \text{and} \quad \sup_{\partial\Omega} (u_\mu - v) = 0.$$

If $\sup_{\Omega} (u_\mu - v) > 0$ then, by Theorem 4.1, there is a point $z \in \Omega$ such that $(u_\mu - v)(p) > 0$ and $g(z, u_\mu(z)) \geq h(z, v(z))$. Since, $v(z) < u_\mu(z)$, $g(z, v(z)) \geq g(z, u_\mu(z)) \geq h(z, v(z))$, a contradiction. Thus the claim holds. \square

PART I

5. PROOF OF THEOREM 1.1 AND A CHANGE OF VARIABLES RESULT UNDER CONDITIONS A AND B

In this section, H satisfies conditions A (monotonicity) and B (homogeneity), see (1.2) and (1.3). We show some additional comparison principles including Theorem 1.1, see [6] and [8]. We also discuss a change of variables result. Recall that $k = k_1 + k_2$ in (1.4).

Proof of Theorem 1.1: Let $U \subset \Omega$ be a compactly contained sub-domain of Ω . We assume that $u > 0$ somewhere in U and show that $\sup_U(u/v) = \sup_{\partial U} u/v$.

Let $p \in U$ is such that $u(p)/v(p) = \sup_U(u/v) > \sup_{\partial U} u/v$. Set $\tau = u(p)/v(p)$. Since $v > 0$, we have $u(p) > 0$ and $\tau > 0$. Thus, the function

$$(5.1) \quad w(x) = u(x) - \tau v(x) \leq 0, \quad x \in \overline{U}.$$

Using (1.2), (1.3), (1.4), (1.14) and $v > 0$, we have that

$$(5.2) \quad H(D\tau v, D^2\tau v) \leq -\tau^k h(x, v(x)) \leq -\tau^{k-m} a(x)(\tau v(x))^m \quad \forall x \in \Omega.$$

From (5.1), $w(x) < 0$, for any $x \in \partial U$, and $\sup_U w = w(p) = 0$. Thus, $\sup_U w > \sup_{\partial U} w$, and applying Theorem 4.1 and (1.14) to u and τv (see (5.2)) there is a $z \in U$ such that

$$(5.3) \quad w(z) = \sup_U w = 0 \text{ and } g(z, u(z)) \geq \tau^k h(z, v(z)) \geq \tau^{k-m} a(z)(\tau v(z))^m > 0.$$

From (5.3) we get $u(z) = \tau v(z) > 0$ and

$$(5.4) \quad g(z, u(z)) \geq \tau^{k-m} a(z)(\tau v(z))^m = \tau^{k-m} a(z)u(z)^m > \tau^{k-m} g(z, u(z)) > 0.$$

We get a contradiction for $k = m$ and part (a)(i) holds. We prove part (a) (ii). Set $\mu_j = \sup_{\partial U_j}(u/v)$. By part(i), μ_j 's are increasing and $\mu = \sup_j \mu_j < \infty$. If $\sup_\Omega(u/v) > \mu$ then there is a set U_j such that $\sup_{U_j}(u/v) > \mu$. This violates part (a)(i) since $\sup_{\partial U_j}(u/v) \leq \mu$.

From (5.4), $\tau^{k-m} < 1$. We get a contradiction for part (b). Thus, the theorem holds. \square

Remark 5.1. Let u, v, g and h be as in the statement of Theorem 1.1. Since $v > 0$, it follows that that if $u > 0$ somewhere in Ω then $u > 0$ somewhere on $\partial\Omega$. As a result, if $u = 0$, on $\partial\Omega$, then $u \leq 0$ in Ω . \square

Remark 5.2. The proof of Theorem 1.1 can be adapted to the case $g(x, t) < a(x)|t|^{k-1}t + f(x) \leq h(x, t)$, $\forall (x, t) \in \Omega \times (0, \infty)$, and where $f(x) \geq 0$. Suppose that $v > 0$ and $u \leq v$ on $\partial\Omega$. If $u - v > 0$ in Ω then there is a $\tau > 1$ such that $\sup_\Omega(u - \tau v) = 0$. Clearly, $u - \tau v < 0$ on $\partial\Omega$. Arguing as in Theorem 1.1 there is a point $p \in \Omega$ such that $(u - \tau v)(p) = \sup_\Omega(u - \tau v) = 0$, and

$$g(p, u(p)) \geq \tau^k h(p, v(p)) \geq a(p)(\tau v(p))^k + \tau^k f(p) \geq a(p)u(p)^k + f(p) > g(p, u(p)).$$

This is a contradiction and $u \leq v$ in Ω . Note the claim holds if we take $0 < m \leq k$. A version may be found in [8] (Theorem 3.6). \square

Next, we extend Theorem 1.1 when the condition (1.14) is relaxed to include the case $g \leq h$.

Lemma 5.3. *Let $a \in C(\Omega)$, $a > 0$, $u \in usc(\overline{\Omega})$, and $v \in lsc(\overline{\Omega}) \cap L^\infty(\Omega)$, $\inf_{\overline{\Omega}} v > 0$. Assume that $g(x, t) \leq a(x)|t|^{k-1}t \leq h(x, t)$, $\forall (x, t) \in \Omega \times (0, \infty)$, where k is as in (1.4). If u, v satisfy*

$$H(Du, D^2u) + g(x, u) \geq 0, \quad \text{and} \quad H(Dv, D^2v) + h(x, v) \leq 0, \quad \text{in } \Omega,$$

and $u > 0$ somewhere in Ω , then $\sup_{\Omega}(u/v) = \sup_{\partial\Omega}(u/v)$.

Proof. Since $\inf_{\Omega} v > 0$, we observe that $h(x, v(x)) \geq a(x)v(x)^k > 0$, $\forall x \in \Omega$. Also, the proof of Lemma 4.2 shows that $v > \inf_{\partial\Omega} v$, in Ω .

Set $\mu = \inf_{\partial\Omega} v$, $\ell = \sup_{\Omega} v$, and $v_\theta = v - \theta\mu$, for $0 < \theta < 1$. Let $\varepsilon > 0$ be small, to be determined. Recalling that $h(x, t) \geq a(x)t^k$, $\forall t \geq 0$, we calculate

$$\begin{aligned} H(Dv_\theta, D^2v_\theta) + (1 + \varepsilon)a(x)v_\theta^k &\leq h(x, v) \left(\frac{(1 + \varepsilon)v_\theta^k}{v^k} - 1 \right) \\ (5.5) \qquad \qquad \qquad &\leq \left((1 + \varepsilon) \left(\frac{\ell - \theta\mu}{\ell} \right)^k - 1 \right) h(x, v). \end{aligned}$$

Note that in $\ell > 0$, $(\ell - \theta\mu)/\ell$ increases in ℓ . We choose ε small enough so that

$$H(Dv_\theta, D^2v_\theta) + (1 + \varepsilon)a(x)v_\theta^k \leq 0, \quad \text{in } \Omega.$$

As done in the proof of Theorem 1.1, we take $\tau = \sup_{\Omega} u/v$ and assume that $\tau > \sup_{\partial\Omega} u/v$. Working with u , v_θ and $w = u - \tau v_\theta$, we obtain that there is a $z \in \Omega$ such that $w(z) = \sup_U w = 0$ and $g(z, u(z)) \geq (1 + \varepsilon)a(z)(\tau v_\theta(z))^k > 0$. Since $\tau v_\theta(z) = u(z)$,

$$g(z, u(z)) \geq (1 + \varepsilon)a(z)u(z)^k \geq (1 + \varepsilon)g(z, u(z)) > 0.$$

This is a contradiction and $\sup_{\Omega}(u/v_\theta) = \sup_{\partial\Omega}(u/v_\theta)$. Letting $\theta \rightarrow 0$ proves the claim. \square

We now present a result regarding a change of variables which will prove useful in Part II.

Lemma 5.4. *Let $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and $\zeta : \mathbb{R} \rightarrow \mathbb{R}$, be a C^2 function with $\zeta' > 0$. Take $u : \Omega \rightarrow \mathbb{R}$ and set $v = \zeta(u)$. Call $\eta = \zeta^{-1}$.*

(a) If ζ is convex and $u \in usc(\Omega)$ satisfies $H(Du, D^2u) + f(x, u) \geq 0$, in Ω , then

$$H(Dv, D^2v) + [\eta'(v(x))]^{-k} f(x, \eta(v(x))) \geq 0, \quad \text{in } \Omega.$$

(b) If ζ is concave and $u \in lsc(\Omega)$ and $H(Du, D^2u) + f(x, u) \leq 0$, in Ω , then

$$H(Dv, D^2v) + [\eta'(v(x))]^{-k} f(x, \eta(v(x))) \leq 0, \quad \text{in } \Omega.$$

Proof. We prove part (a). Clearly, $v \in usc(\Omega)$. Since ζ is convex, we have

$$(5.6) \qquad \qquad \qquad \zeta(t_2) - \zeta(t_1) \geq \zeta'(t_1)(t_2 - t_1).$$

Let $\psi \in C^2(\Omega)$ and $p \in \Omega$ be such that $v(x) - \psi(x)$ has a maximum at p , i.e., $(v - \psi)(x) \leq (v - \psi)(p)$. Thus, $\zeta'(u(p))(u(x) - u(p)) \leq \zeta(u(x)) - \zeta(u(p)) \leq \psi(x) - \psi(p)$, see (5.6). Rearranging,

$$(5.7) \quad u(x) - \frac{\psi(x)}{\zeta'(u(p))} \leq u(p) - \frac{\psi(p)}{\zeta'(u(p))}.$$

Since u is a sub-solution, we obtain

$$H\left(\frac{D\psi(p)}{\zeta'(u(p))}, \frac{D^2\psi(p)}{\zeta'(u(p))}\right) + f(p, u(p)) \geq 0.$$

Using $\zeta'(\eta(t))\eta'(t) = 1$ together with (1.2), (1.3) and (1.4) we rewrite the above as

$$H(D\psi(p), D^2\psi(p)) + [\eta'(v(p))]^{-k} f(p, \eta(v(p))) \geq 0.$$

To prove part (b), we observe that the inequalities in (5.6) and (5.7) are reversed. One may now argue similarly to show part (b). \square

Remark 5.5. Let g and h , in $C(\Omega, \mathbb{R})$, be such that $g(x, t) \leq a(x)t^k \leq h(x, t), \forall (x, t) \in \Omega \times \mathbb{R}^+$. Suppose that $u : \Omega \rightarrow \mathbb{R}^+$ and $\zeta(t) = t^\beta, t \geq 0$. Lemma 5.4 implies the following.

(i) Let $u \in usc(\Omega)$ solve $H(Du, D^2u) + g(x, u) \geq 0$, in Ω . If $\beta > 1$ then $H(Dv, D^2v) + \beta^k a(x)v^k \geq 0$.

(ii) Let $u \in lsc(\Omega)$ solve $H(Du, D^2u) + h(x, u) \leq 0$, in Ω . If $\beta < 1$ then $H(Dv, D^2v) + \beta^k a(x)v^k \leq 0$. \square

Next, Theorem 1.1, Lemma 5.3 and Remark 5.5 imply the following comparison principle.

Lemma 5.6. Let g, h be in $C(\Omega, \mathbb{R})$ and $g(x, t) \leq a(x)|t|^{k-1}t \leq h(x, t), \forall (x, t) \in \Omega \times (0, \infty)$, where $a \in C(\Omega)$ and $a > 0$.

Suppose that (i) $0 < \beta < 1$ is such that $g(x, t) \leq \beta^k a(x)t^k, \forall t \geq 0$, and (ii) $\sup_{x \in \Omega} |h(x, t)| = 0$ iff $t = 0$. Let $u \in usc(\overline{\Omega})$ and $v \in lsc(\overline{\Omega}), \inf_{\overline{\Omega}} v > 0$, solve

$$H(Du, D^2u) + g(x, u) \geq 0, \text{ and } H(Dv, D^2v) + h(x, v) \leq 0, \text{ in } \Omega.$$

Assume that $u > 0$ somewhere in Ω . If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω . Also,

$$(5.8) \quad \sup_{\Omega} \frac{u}{v^\beta} \leq \sup_{\partial\Omega} \frac{u}{v^\beta}.$$

If $0 < \sup_{\partial\Omega} u \leq \inf_{\partial\Omega} v$, on $\partial\Omega$, then $u < v$, in Ω , and $u(z) \leq (\inf_{\partial\Omega} v)^{1-\beta} v(z)^\beta, \forall z \in \Omega$.

Proof. By Theorem 1.1 and Lemma 5.3, if $u \leq v$ on $\partial\Omega$ then $u \leq v$, in Ω . By Remark 5.5 and the lower bound for h , $w = v^\beta$ solves $H(Dw, D^2w) + \beta^k a(x)w^k \leq 0$. From Lemma 5.3 and the upper bound for g it is seen that (5.8) holds. By Lemma 4.2, $v > \inf_{\partial\Omega} v$. Next, using $u \leq v$ in Ω , we get for any $z \in \Omega$,

$$\frac{u(z)}{v(z)^\beta} \leq \sup_{\partial\Omega} \left(\frac{u}{v^\beta}\right) \leq \frac{\sup_{\partial\Omega} u}{\inf_{\partial\Omega} v^\beta} \leq \frac{\inf_{\partial\Omega} v}{\inf_{\partial\Omega} v^\beta} \leq \inf_{\partial\Omega} v^{1-\beta} < v(z)^{1-\beta}.$$

Thus, $u(z) < v(z)$. The above inequality also implies $u(x) \leq \inf_{\partial\Omega} v^{1-\beta} v^\beta(x), \forall x \in \Omega$. \square

6. A PRIORI BOUNDS: PROOF OF THEOREM 1.2

In this section, we derive some useful a priori bounds for a fairly general class of functions $f(x, t)$. We assume that H satisfies conditions A, B and C, see (1.2)-(1.10). However, we make no use of (1.10) in this section.

We state the following version of the maximum principle, see Lemma 4.2 in this context.

Lemma 6.1. (*Maximum principle*)

(i) If $u \in usc(\overline{\Omega})$ solves $H(Du, D^2u) \geq 0$, in Ω , then $\sup_{\Omega} u = \sup_{\partial\Omega} u$. (ii) If $u \in lsc(\overline{\Omega})$ solves $H(Du, D^2u) \leq 0$, in Ω , then $\inf_{\Omega} u = \inf_{\partial\Omega} u$.

Proof. Let $q \in \mathbb{R}^n \setminus \overline{\Omega}$ and $0 < \rho < R < \infty$ be such that $\Omega \subset B_R(q) \setminus B_\rho(q)$. We prove (i) by contradiction (part (ii) is similar). Let $\varepsilon > 0$ and $p \in \Omega$ be such that $u(p) \geq \sup_{\partial\Omega} u + \varepsilon$. Define

$$w(x) = \sup_{\partial\Omega} u + \frac{\varepsilon}{2} \left(\frac{R^2 - |x - q|^2}{R^2 - \rho^2} \right), \quad \forall x \in B_R(q) \setminus B_\rho(q).$$

Thus, $\sup_{\partial\Omega} u \leq w(x) \leq \sup_{\partial\Omega} u + \varepsilon/2$, in $B_R(q) \setminus B_\rho(q)$. Clearly, $u(p) - w(p) > 0$ and $u - w \leq 0$, on $\partial\Omega$. Let $z \in \Omega$ be a point of maximum of $u - w$ on $\overline{\Omega}$. By (1.2), (1.3), (1.6), (1.8) and (1.9),

$$H(Dw(z), D^2w(z)) = \left(\frac{\varepsilon}{R^2 - \rho^2} \right)^k |x - q|^{k_1} H(e, -I) \leq -m_1 \rho^{k_1} \left(\frac{\varepsilon}{R^2 - \rho^2} \right)^k < 0,$$

where e is a unit vector in \mathbb{R}^n and $m_1 \leq m_1(0)$. We get a contradiction and the claim holds. \square

Remark 6.2. Let $u^\pm(x) = c \pm d|x|^\alpha$, $d > 0$ and $\alpha = 1 + k_2/k$, see (1.6), (1.8) and (1.9).

We remark that (2.8) holds in the viscosity sense at $r = 0$. Suppose that $\psi \in C^2$ is such that $(u^+ - \psi)(x) \leq (u^+ - \psi)(o)$. Then $d|x|^\alpha \leq \langle D\psi(o), x \rangle + o(|x|)$ as $x \rightarrow 0$. If $x = -\varepsilon D\psi(o)$, $\varepsilon > 0$, we get $d\varepsilon^{\alpha-1} \leq -|D\psi(o)|^{2-\alpha} + o(1)$, as $\varepsilon \rightarrow 0$. Thus, $D\psi(o) = 0$ and $d|x|^\alpha \leq \langle D^2\psi(o)x, x \rangle/2 + o(|x|^2)$, as $x \rightarrow o$. Since $1 < \alpha < 2$, this is a contradiction. The inequality $H(Du^+, D^2u^+) \geq (\alpha d)^k m_1 > 0$ holds, see (2.8). Next, let $\phi \in C^2$ be such that $(u^- - \phi)(x) \geq (u^- - \phi)(o)$. Then, $-d|x|^\alpha \geq \langle D\phi(o), x \rangle + o(|x|)$ and $D\phi(o) = 0$. Thus, $-d|x|^\alpha \geq \langle D^2\phi(o)x, x \rangle/2 + o(|x|^2)$, as $|x| \rightarrow 0$. This is a contradiction and $H(Du^-, D^2u^-) \leq -(\alpha d)^k m_1$, see (1.6), (1.8) and (2.8). \square

We consider the problem

$$(6.1) \quad H(Du, D^2u) + f(x, u) = 0, \quad \text{in } \Omega \quad \text{and } u = h \text{ on } \partial\Omega,$$

where $h \in C(\partial\Omega)$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$. For a function g , define $g^+ = \max\{g, 0\}$ and $g^- = \min\{g, 0\}$. We now present a priori supremum bounds when $f(x, u) = f(x)$, also see [7].

Lemma 6.3. Let $f \in C(\Omega) \cap L^\infty(\Omega)$, $h \in C(\partial\Omega)$, α and σ be as in (1.9). Suppose that $B_{R_o}(z_o)$, for some $z_o \in \mathbb{R}^n$, is the out-ball of Ω . Consider the problem

$$(6.2) \quad H(Du, D^2u) + f(x) = 0, \quad \forall x \in \Omega, \quad u = h, \quad \text{on } \partial\Omega,$$

(i) If $u \in usc(\overline{\Omega})$ is a sub-solution of (6.3) then $\sup_{\Omega} u \leq \sup_{\partial\Omega} h + \sigma(\sup_{\Omega} f^+)^{1/k} R_o^\alpha$.

(ii) Similarly, if $u \in \text{lsc}(\overline{\Omega})$ is a super-solution of (6.3) then $\inf_{\Omega} u \geq \inf_{\partial\Omega} h - \sigma |\inf_{\Omega} f^-|^{1/k} R_o^\alpha$.

Proof. We prove part (i). Let u be a sub-solution of (6.2). Fix $\varepsilon > 0$ and consider the function

$$w_\varepsilon(x) = \sup_{\partial\Omega} h + \sigma(\sup_{\Omega} f^+ + \varepsilon)^{1/k} (R_o^\alpha - |x - z_0|^\alpha), \forall x \in \overline{\Omega}.$$

Applying (1.3), (1.4), (1.6), (1.8), (1.9), (2.8) and Remark 6.2, we have

$$H(Dw_\varepsilon, D^2w_\varepsilon) = \left(\frac{\sup_{\Omega} f^+ + \varepsilon}{m_1} \right) H \left(e, \frac{k_1}{k} e \otimes e - I \right) \leq -\sup_{\Omega} f^+ - \varepsilon < -f, \text{ in } \Omega.$$

Also, $w_\varepsilon \geq h$, on $\partial\Omega$. Thus, Lemma 4.3 implies $u(x) \leq w_\varepsilon(x)$, in Ω . Since ε is arbitrary the claim follows.

Part (ii) follows by taking $\hat{w}_\varepsilon(x) = \inf_{\Omega} h - \sigma(|\inf_{\Omega} f^-| + \varepsilon)^{1/k} (R_o^\alpha - |x - z_0|^\alpha)$, $\forall x \in \overline{\Omega}$. \square

Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy

$$(6.3) \quad \sup_{\Omega \times [t_1, t_2]} |f(x, t)| < \infty, \quad \forall t_1, t_2 \text{ such that } -\infty < t_1 \leq t_2 < \infty.$$

We apply Lemma 6.3 to prove Theorem 1.2. A related result is proven in [7].

Proof of Theorem 1.2. Set $M = \sup_{\Omega} |u|$, $L = \sup_{\partial\Omega} |h|$ and R_o the radius of the out-ball of Ω . Let $\varepsilon > 0$, small, be fixed.

We prove part (a). By (1.16) there exists $t_1 > 0$ such that

$$(6.4) \quad (\mu_1 - \varepsilon)|t|^k \leq \inf_{\Omega} f(x, t) \leq \sup_{\Omega} f(x, t) \leq (\mu_2 + \varepsilon)|t|^k, \quad \forall |t| > t_1.$$

By (6.3), there is a $0 < \mu_3 < \infty$ such that $\sup_{[-t_1, t_1] \times \Omega} |f(x, t)| \leq \mu_3$. Define $s_4 = \max(|\mu_1| + \varepsilon, \mu_2 + \varepsilon)$, then $\sup_{\Omega} |f(x, t)| \leq \mu_4 |t|^k + \mu_3$, $-\infty < t < \infty$.

Thus, we have $|f(x, u)| \leq \mu_4 M^k + \mu_3$ and

$$(6.5) \quad -|\lambda|(\mu_4 M^k + \mu_3) \leq H(Du, D^2u) \leq |\lambda|(\mu_4 M^k + \mu_3), \text{ in } \Omega.$$

Using $-L \leq u \leq L$ on $\partial\Omega$ and applying the estimates of Lemma 6.3 to (6.5) we have

$$0 \leq |u| \leq M \leq L + \sigma |\lambda|^{1/k} (\mu_4 M^k + \mu_3)^{1/k} R_o^\alpha \leq L + |\lambda|^{1/k} \mu_5 M + \mu_6.$$

where $\mu_5 > 0$ and $\mu_6 > 0$ are independent of M . Hence, $M \leq (L + \mu_6)(1 - |\lambda|^{1/k} \mu_5)^{-1}$. It is clear that if $|\lambda|$ is small enough u is a priori bounded.

To show (b), take $\varepsilon = \{|\lambda|(2\sigma R_o^\alpha)^k\}^{-1}$, and we get from (6.4), $|f(x, t)| \leq \varepsilon |t|^k$, $|t| \geq t_1 > 0$, where $t_1 > L$. Suppose that $M > t_1$, then $H(Du, D^2u) \leq \varepsilon \lambda M^k$ in the set $\{u > t_1\}$. Using Lemma 6.3, $M \leq t_1 + \sigma(\varepsilon \lambda)^{1/k} M$. Applying the definition of ε , $\sup_{\Omega} u \leq 2t_1$. A similar argument can be used to obtain a lower bound for $\inf_{\Omega} u$. \square

Part II

7. ESTIMATES FOR THE EIGENVALUE PROBLEM. PROOF OF THEOREM 1.3.

In Part II, H satisfies conditions A , B and C , see (1.2)-(1.9). From hereon, $\lambda \in \mathbb{R}$ stands for a parameter, $a \in C(\Omega, \mathbb{R})$ and $\delta \geq 0$. Assume that there are $0 < \mu \leq \nu < \infty$ such that

$$(7.1) \quad 0 < \mu \leq a(x) \leq \nu < \infty, \quad \forall x \in \Omega.$$

We study the problem

$$(7.2) \quad H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega \text{ and } u = h \text{ on } \partial\Omega,$$

where $u \in C(\overline{\Omega})$, $h \in C(\partial\Omega)$ and $\inf_{\partial\Omega} h > 0$.

We record an observation for (7.2) and include a comment relevant to the eigenvalue problem for H when H is odd in X .

Remark 7.1. (i) Let $\lambda > 0$, $a(x)$ be as in (7.1) and $h \in C(\partial\Omega)$, $\inf_{\Omega} h > 0$. If $u \in C(\overline{\Omega})$, $u > 0$, solves

$$H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega \text{ and } u = h \text{ on } \partial\Omega,$$

then $u > \inf_{\partial\Omega} h$ in Ω and is unique. These follow from Lemmas 4.2 and 5.3. We will show in Lemma 7.2 that for small $\lambda > 0$ any solution u is necessarily positive.

(ii) Suppose that $H(p, -X) = -H(p, X)$, $\forall (p, X) \in \mathbb{R}^n \times S^n$. We show that if (7.2) has a positive super-solution for some $\lambda > 0$ then any solution of (7.2) is necessarily positive. Hence, if, for some $\lambda > 0$, a solution changes sign, then (7.2) has no positive super-solutions.

Let $v \in C(\overline{\Omega})$, $v > 0$, solve $H(Dv, D^2v) + \lambda a(x)v^k \leq 0$, in Ω , and $v \geq h$ on $\partial\Omega$. Let u be any solution of (7.2) that changes sign in Ω . Set $\Omega^- = \{u < 0\}$. Take $w = -u$, then $w > 0$, in Ω^- , $H(Dw, D^2w) + \lambda a(x)w^k = 0$, in Ω^- , and $w = 0$, on $\partial\Omega^-$. Use Lemma 5.3 in every component of Ω^- to conclude that $\sup_{\Omega^-} w/v \leq \sup_{\partial\Omega^-} (w/v) = 0$. Thus, $\Omega^- = \emptyset$ and $u \geq 0$. Lemma 4.2 yields that $v > \inf_{\partial\Omega} h$. Uniqueness follows from Lemma 5.3. \square

Next, recalling (7.1) and applying the estimates of Lemma 6.3 to a solution u of (7.2), we get

$$\inf_{\partial\Omega} h - \sigma|\nu\lambda|^{1/k}R_o^\alpha \inf_{\Omega} u^- \leq u(x) \leq \sup_{\partial\Omega} h + \sigma|\nu\lambda|^{1/k}R_o^\alpha \sup_{\Omega} u^+.$$

where α and σ are as in (1.9). Setting $\Lambda = \nu^{-1}(\sigma R_o^\alpha)^{-k}$, we obtain

$$(7.3) \quad \inf_{\partial\Omega} h - \left(\frac{|\lambda|}{\Lambda}\right)^{1/k} \inf_{\Omega} u^- \leq u(x) \leq \sup_{\partial\Omega} h + \left(\frac{|\lambda|}{\Lambda}\right)^{1/k} \sup_{\Omega} u^+,$$

Our next result discusses the influence of λ on the solutions of (7.2).

Lemma 7.2. Let $a \in C(\Omega)$ be as in (7.1) and Λ be as in (7.3). Suppose that $u \in C(\overline{\Omega})$ solves

$$H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega, \text{ and } u = h \text{ on } \partial\Omega,$$

where $h \in C(\partial\Omega)$. Set $\kappa_1 = \inf_{\partial\Omega} h$ and $\kappa_2 = \sup_{\partial\Omega} h$. Then the following hold.

(i) If $\lambda \leq 0$ then $\min(0, \kappa_1) \leq u \leq \max(0, \kappa_2)$ in Ω . If $\lambda = 0$ then $\kappa_1 \leq u \leq \kappa_2$ in Ω .

(ii) If $h = 0$ and u is a non-zero solution then $\lambda > 0$. (iii) If $0 < \lambda < \Lambda$ then

$$\frac{\kappa_1}{1 - (\lambda/\Lambda)^{1/k}} \leq \inf_{\Omega} u^- \leq u(x) \leq \sup_{\Omega} u^+ \leq \frac{\kappa_2}{1 - (\lambda/\Lambda)^{1/k}}.$$

In particular, if $h \geq 0$ then $\kappa_1 \leq u \leq \theta \kappa_2$, where $\theta = (1 - (\lambda/\Lambda)^{1/k})^{-1}$. Thus, if $\lambda > 0$ is small and $h > 0$ then any solution u is positive in Ω .

Proof. We use Lemma 6.1. We prove part (i). Let $\lambda \leq 0$ and $\Omega^- = \{x \in \Omega : u(x) < \min(0, \kappa_1)\}$ be non-empty. Then $H(Du, D^2u) \leq 0$, in Ω^- , and this contradicts Lemma 6.1(ii). Next, if $\Omega^+ = \{x \in \Omega : u(x) > \max(0, \kappa_2)\}$ is non-empty then $H(Du, D^2u) \geq 0$ in Ω^+ . This contradicts Lemma 6.1 (i). Part (ii) now follows as a contrapositive of (i). To show (iii), we use (7.3) and conclude that

$$\frac{\inf_{\partial\Omega} h}{1 - (\lambda/\Lambda)^{1/k}} \leq \inf_{\Omega} u^- \leq u(x) \leq \sup_{\Omega} u^+ \leq \frac{\sup_{\partial\Omega} h}{1 - (\lambda/\Lambda)^{1/k}}.$$

If $\inf_{\partial\Omega} h \geq 0$ then $\inf_{\Omega} u^- = 0$ and we obtain the final estimate in the lemma. \square

We now show that if (7.2) has a positive super-solution then it has a super-solution for a slightly larger value of λ and a sub-solution for a smaller value of λ .

Theorem 7.3. *Let $a \in C(\Omega)$ be as in (7.1), $h \in C(\partial\Omega)$, $\inf_{\partial\Omega} h > 0$, and $\lambda > 0$. Suppose that $u \in L^\infty(\Omega)$ and $u > 0$. Set $\vartheta = \inf_{\partial\Omega} h$; define $v_\theta = (u - \theta\vartheta)/(1 - \theta)$, $\forall 0 \leq \theta < 1$, and $w_\theta = (u + \theta\vartheta)/(1 + \theta)$, $\forall \theta > 0$.*

(i) *If $u \in \text{lsc}(\overline{\Omega})$ solves $H(Du, D^2u) + \lambda a(x)u^k \leq 0$, in Ω , and $u \geq h$ on $\partial\Omega$, then, for every*

$$0 < \varepsilon \leq \theta \lambda k \left(\frac{(\vartheta / \sup_{\Omega} u)}{1 - \theta(\vartheta / \sup_{\Omega} u)} \right), \quad 0 \leq \theta < 1,$$

the function v_θ solves $H(Dv_\theta, D^2v_\theta) + (\lambda + \varepsilon)a(x)v_\theta^k \leq 0$, in Ω , and $v_\theta \geq h$ on $\partial\Omega$.

(ii) *Suppose that $u \in \text{usc}(\overline{\Omega})$ solves $H(Du, D^2u) + \lambda a(x)u^k \geq 0$, in Ω , and $u \leq h$ on $\partial\Omega$. For every $0 < \varepsilon < \lambda$ there is a $\theta > 0$ such that w_θ solves*

$$H(Dw_\theta, D^2w_\theta) + (\lambda - \varepsilon)a(x)w_\theta^k \geq 0, \quad \text{in } \Omega, \text{ and } w_\theta \leq h \text{ on } \partial\Omega.$$

Proof. Set $m = \sup_{\Omega} u$.

(i) Fix $0 < \theta \leq 1$ and set $\eta = u - \theta\vartheta$. By Lemma 6.1, $u \geq \vartheta$, and $\eta \geq (1 - \theta)\vartheta$, in Ω , and $\eta \geq (1 - \theta)h$ on $\partial\Omega$. Observe that $(t - \theta\vartheta)/t$, $t \geq \vartheta$, is increasing in t . Calculating,

$$\begin{aligned} H(D\eta, D^2\eta) + (\lambda + \varepsilon)a(x)\eta^k &\leq a(x) \left\{ (\lambda + \varepsilon)\eta^k - \lambda u^k \right\} = a(x)u^k \left\{ (\lambda + \varepsilon) \left(\frac{u - \theta\vartheta}{u} \right)^k - \lambda \right\} \\ &\leq a(x)u^k \left\{ (\lambda + \varepsilon) \left(\frac{m - \theta\vartheta}{m} \right)^k - \lambda \right\} \leq 0, \end{aligned}$$

if we choose $0 < \varepsilon \leq \lambda \{m^k(m - \theta\vartheta)^{-k} - 1\}$. Using the lower bound $k(t - 1) \leq t^k - 1$, $t \geq 1$, we take

$$0 < \varepsilon \leq \theta \lambda k \left(\frac{(\vartheta/m)}{1 - \theta(\vartheta/m)} \right).$$

Using the homogeneity of H , $v_\theta(x) = \eta/(1 - \theta) = (u - \theta\vartheta)/(1 - \theta)$, $0 \leq \theta < 1$, $\forall x \in \Omega$, solves $H(Dv_\theta, D^2v_\theta) + (\lambda + \varepsilon)a(x)v_\theta^k \leq 0$, in Ω , and $v_\theta \geq h$, on $\partial\Omega$.

(ii) Let $0 < \varepsilon < \lambda$ be fixed and $\theta > 0$ to be determined. Set $\varphi = u + \theta\vartheta$, in $\overline{\Omega}$ and calculate to obtain

$$\begin{aligned} H(D\varphi, D^2\varphi) + (\lambda - \varepsilon)a(x)\varphi^k &\geq a(x)u^k \left((\lambda - \varepsilon)\frac{\varphi^k}{u^k} - \lambda \right) = a(x)u^k \left((\lambda - \varepsilon) \left(\frac{u + \theta\vartheta}{u} \right)^k - \lambda \right) \\ &\geq a(x)u^k \left((\lambda - \varepsilon) \left(\frac{m + \theta\vartheta}{m} \right)^k - \lambda \right) \geq 0, \end{aligned}$$

if θ is such that $0 < \varepsilon \leq \lambda \{ (m + \theta\vartheta)^k m^{-k} - 1 \}$. Clearly, $w_\theta \geq \vartheta$, in Ω , and $w_\theta \leq h$ on $\partial\Omega$. \square

We introduce a quantity that will be useful for the eigenvalue problem. Let $\delta > 0$ and $\lambda > 0$. Consider the problem of finding a positive solution $u_\lambda \in C(\overline{\Omega})$ of

$$(7.4) \quad H(Du_\lambda, D^2u_\lambda) + \lambda a(x)|u_\lambda|^{k-1}u_\lambda = 0, \text{ in } \Omega, \text{ and } u_\lambda = \delta \text{ on } \partial\Omega.$$

Define

$$(7.5) \quad \lambda_\Omega = \sup\{\lambda : (7.4) \text{ has a positive solution } u_\lambda\}.$$

We will show in Sections 8 and 9 that $0 < \lambda_\Omega < \infty$. For the next result we assume this fact.

Theorem 7.4. *Suppose that $\lambda_\Omega > 0$, where λ_Ω is as in (7.5) and $0 < \lambda < \lambda_\Omega$. Let $a \in C(\Omega)$ be as in (7.1) and $u_\lambda > 0$ be a solution of (7.4). Then the following hold.*

(i) *The solution u_λ is unique and $u_\lambda > \delta$. (ii) For every $x \in \Omega$, the function $u_\lambda(x)$ increases as λ increases. (iii) Call $m_\lambda = \sup_\Omega u_\lambda$. If $0 < \lambda_\Omega < \infty$ then*

$$m_\lambda \geq \delta \left(1 + \frac{k\lambda}{\lambda_\Omega - \lambda} \right), \quad 0 < \lambda < \lambda_\Omega.$$

Thus, $m_\lambda \rightarrow \infty$ as $\lambda \rightarrow \lambda_\Omega$.

(iv) *The set of λ 's for which (7.4) has a positive solution is the interval $[0, \lambda_\Omega)$.*

Proof. Parts (i) and (ii) follow from Remark 7.1 and Theorem 1.1. We use Theorem 7.3 (i) to prove part (iii). To see this, let u_λ be the solution of (7.4) for some $\lambda < \lambda_\Omega$. Let $0 < \theta < 1$ and ε be as in part (i) of Theorem 7.3. Then

$$0 < \varepsilon \leq \theta\lambda k \left(\frac{(\delta/m_\lambda)}{1 - \theta(\delta/m_\lambda)} \right) \leq \lambda k \left(\frac{(\delta/m_\lambda)}{1 - (\delta/m_\lambda)} \right), \quad \forall 0 < \theta < 1.$$

Clearly, $\lambda + \varepsilon \leq \lambda_\Omega$. Letting $\theta \uparrow 1$,

$$\lambda_\Omega - \lambda \geq \lambda k \left(\frac{(\delta/m_\lambda)}{1 - (\delta/m_\lambda)} \right).$$

Rearranging, we obtain the estimate in part (iii).

For part (iv), let $\lambda < \lambda_\Omega$ and (7.4) have a solution u_λ . If $0 < \hat{\lambda} < \lambda$ then $H(Du_\lambda, D^2u) + \hat{\lambda}a(x)u_\lambda(x)^k \leq 0$, with $u_\lambda = \delta$ on $\partial\Omega$. Thus, u_λ is super-solution and $v = \delta$ is a sub-solution. By

Lemma 5.3 and Remark 2.1, the problem $H(Dw, D^2w) + \hat{\lambda}a(x)w^k = 0$, in Ω , with $w = \delta$, has a positive solution.

From Theorem 7.3(i), v_θ is a super-solution of (7.4) with $\lambda + \varepsilon$ and $v_\theta = \delta$ on $\partial\Omega$. Also, the function $v = \delta$ is a sub-solution in Ω . By Lemma 5.3 and Remark 2.1 there is a positive solution of $H(Dw, D^2w) + (\lambda + \varepsilon)a(x)w^k = 0$, in Ω , and $w = \delta$ on $\partial\Omega$. \square

Theorem 7.3 will be instrumental for proving Theorem 1.3. We now show that u_λ , a solution of (7.4), is an increasing Lipschitz continuous function of λ , for $0 < \lambda < \lambda_\Omega$.

Proof of Theorem 1.3.

Proof. By Theorem 7.4 (iv), λ is an interior point. Fix λ and $x \in \Omega$. Thus, (1.19) has a unique positive solution u_λ and by Theorem 1.1, $v_x(\lambda) = u_\lambda(x)$ is non-decreasing in λ . Set $M_\lambda = \sup u_\lambda$.

We make repeated use of Theorem 7.3 (i). Recall that if u_λ solves (1.19) then for $0 < \theta < 1$ and $0 < \varepsilon \leq \theta k \lambda (\delta/M_\lambda)$ there is a solution $u_{\lambda+\varepsilon} > 0$ of

$$(7.6) \quad H(Du_{\lambda+\varepsilon}, D^2u_{\lambda+\varepsilon}) + (\lambda + \varepsilon)a(x)u_{\lambda+\varepsilon}^k = 0, \text{ in } \Omega, \text{ and } u_{\lambda+\varepsilon} = \delta.$$

Also, $w(x) = (u_\lambda - \theta\delta)(1 - \theta)^{-1}$ is a super-solution of (7.6), and by Lemma 5.3,

$$(7.7) \quad u_{\lambda+\varepsilon} \leq w, \text{ in } \Omega.$$

Upper Bound By (7.6) and (7.7), for $0 < \theta < 1$ and $0 < \varepsilon \leq \theta k \lambda (\delta/M_\lambda)$, we have

$$v_x(\lambda + \varepsilon) = u_{\lambda+\varepsilon}(x) \leq \frac{u_\lambda(x) - \theta\delta}{1 - \theta} = \frac{v_x(\lambda) - \theta\delta}{1 - \theta}.$$

Taking $\varepsilon = k\theta\lambda_0(\delta/M_\lambda)$, θ small, we get

$$0 \leq \frac{v_x(\lambda + \varepsilon) - v_x(\lambda)}{\varepsilon} \leq \frac{\theta(v_x(\lambda) - \delta)}{(1 - \theta)\varepsilon} = \left(\frac{M_\lambda}{k\delta}\right) \frac{v_x(\lambda) - \delta}{\lambda(1 - \theta)}$$

Thus, the right hand derivative $D_\lambda^+ v_x(\lambda) \leq (M_\lambda/k\delta)(v_x(\lambda) - \delta)/\lambda$, by letting $\theta \rightarrow 0$.

We now compute the left hand derivative $D_\lambda^- M_\lambda$ as follows. Fix $0 < \theta < 1$, small, and choose $\hat{\lambda} < \lambda$ such that (see (7.6) and (7.7))

$$(7.8) \quad \hat{\lambda} = \lambda \left(1 - k\theta \left(\frac{\delta}{M_{\hat{\lambda}}}\right)\right) \quad \text{and} \quad \varepsilon = k\theta \left(\frac{\delta\hat{\lambda}}{M_{\hat{\lambda}}}\right)$$

Thus, $\lambda = \hat{\lambda} + \varepsilon$. Observe that $v_x(\hat{\lambda}) \leq v_x(\lambda)$ and $M_{\hat{\lambda}} \leq M_\lambda$.

Using (7.7), (7.8), Theorem 7.4 and Lemma 5.3 yield $v_x(\hat{\lambda} + \varepsilon) = v_x(\lambda) \leq (v_x(\hat{\lambda}) - \theta\delta)/(1 - \theta)$, and

$$0 \leq \frac{v_x(\hat{\lambda} + \varepsilon) - v_x(\hat{\lambda})}{\varepsilon} = \frac{v_x(\lambda) - v_x(\hat{\lambda})}{\varepsilon} \leq \frac{\theta(v_x(\hat{\lambda}) - \theta\delta)}{\varepsilon(1 - \theta)} \leq \left(\frac{M_\lambda}{k\delta}\right) \frac{v_x(\lambda) - \delta}{\hat{\lambda}(1 - \theta)}.$$

Letting $\theta \rightarrow 0$, we get $D_\lambda^- v_x(\lambda) \leq (M_\lambda/k\delta)(v_x(\lambda) - \delta)/\lambda$. Clearly, $v_x(\lambda)$ is Lipschitz continuous for fixed x and $\delta > 0$. The upper bound in the theorem holds.

Lower Bound. Let $0 < \lambda_1 < \lambda_2 < \lambda_\Omega$. Using (1.19), Remark 5.5 and Lemma 5.6,

$$v_x(\lambda_1)/\delta \leq (v_x(\lambda_2)/\delta)^\tau, \text{ where } \tau = (\lambda_1/\lambda_2)^{1/k} < 1.$$

We obtain $\log(v_x(\lambda_1)/\delta) \leq \tau \log(v_x(\lambda_2)/\delta)$. Subtracting $\tau \log(v_x(\lambda)/\delta)$ from both sides, rearranging and noting that $v_x(\lambda)$ is Lipschitz continuous, we see that

$$\frac{\log(v_x(\lambda_2)/\delta) - \log(v_x(\lambda_1)/\delta)}{\lambda_2 - \lambda_1} \geq \left(\frac{\lambda_2^{1/k} - \lambda_1^{1/k}}{\lambda_1^{1/k}(\lambda_2 - \lambda_1)} \right) \log(v_x(\lambda_1)/\delta).$$

The conclusion follows by letting $\lambda_2 \rightarrow \lambda_1$. \square

Remark 7.5. The above theorem holds if $v_x(\lambda)$ is replaced by $M_\lambda = \sup v_x(\lambda)$. \square

8. EXISTENCE: PROOFS OF THEOREMS 1.4 AND 1.5

We now present the proof of Theorem 1.4 and show the existence of a solution of (7.2), for small $\lambda > 0$ which in turn will imply that $\lambda_\Omega > 0$. This is done by constructing suitable sub-solutions and super-solutions. The two cases in (1.10) are addressed separately.

Proof of Theorem 1.4. Set

$$\nu = \sup_{x \in \Omega} a(x), \quad m = \inf_{\partial\Omega} h, \quad \bar{m} = \sup_{\partial\Omega} h, \quad R = \text{diam}(\Omega), \text{ and assume that } m > 0 \text{ and } \nu < \infty.$$

Using (2.6) and (2.7), we construct suitable sub-solutions and super-solutions to achieve our goal.

Fix $y \in \partial\Omega$ and $\varepsilon > 0$, small, be such that $m - 2\varepsilon > 0$. Let $r = |x - y|$. By continuity, there is a $\eta > 0$ such that

$$(8.1) \quad h(y) - \varepsilon \leq h(x) \leq h(y) + \varepsilon, \quad \forall x \in \overline{B}_\eta(y) \cap \partial\Omega.$$

Case (a): Suppose that (1.10)(i) holds. Fix $\beta = 2 - \bar{s}$. Let $v^\pm = c \pm dr^\beta$ where $c > 0$ and $d > 0$. By (1.8) and (2.6),

$$(8.2) \quad H(Dv^+, D^2v^+) \leq \frac{(d\beta)^k}{r^{\gamma-k\beta}} m_2(\bar{s}) < 0, \quad \text{and} \quad H(Dv^-, D^2v^-) \geq \frac{(d\beta)^k}{r^{\gamma-k\beta}} |m_2(\bar{s})| > 0.$$

Note that $0 < \beta < 1$, $k = k_1 + k_2$, $\gamma = k_1 + 2k_2$ and $\gamma - k\beta > 0$, see (1.9).

We construct a sub-solution v . We assume that $h(y) > m$, otherwise we take $v(x) = m$ in $\overline{\Omega}$. Note that $h(y) - \varepsilon - (m/2) > 0$. Set $r = |x - y|$ and

$$(8.3) \quad v^-(x) = \begin{cases} h(y) - \varepsilon - (h(y) - \varepsilon - m/2)(r/\eta)^\beta, & \text{in } \overline{B}_\eta(y) \cap \overline{\Omega}, \\ m/2, & \text{in } \overline{\Omega} \setminus \overline{B}_\eta(y). \end{cases}$$

Then $v^-(y) = h(y) - \varepsilon$, $v^- = m/2$, on $\partial B_\eta(y)$, and $m/2 \leq v^-(x) \leq h(y) - \varepsilon$, in Ω .

Applying (8.2), we obtain that $H(Dv^-, D^2v^-) \geq 0$ and v^- is a sub-solution of (1.21) in $B_\eta(y) \cap \Omega$. Also, v^- is a sub-solution in $\Omega \setminus \overline{B}_\eta(y)$ and by (8.1) $v^- \leq h$ on $\partial\Omega$. To show that v^- is a sub-solution in Ω , let $p \in \partial B_\eta(y) \cap \Omega$ and $\psi \in C^2$ be such that $(v^- - \psi)(x) \leq (v^- - \psi)(p)$. Since $v^-(p) = m/2$ and $v^-(x) \geq v^-(p)$, we get $0 \leq \langle D\psi(p), x - p \rangle + o(|x - p|)$ as $x \rightarrow p$.

It follows that $D\psi(p) = 0$ and a second order expansion shows that $D^2\psi(p) \geq 0$. Clearly, $H(D\psi(p), D^2\psi(p)) + \lambda a(p)(v^-(p))^k \geq 0$.

Next, we construct a super-solution v^+ . We assume that $h(y) < \bar{m}$, otherwise take $v^+(x) = \bar{m}$ in Ω . For a fixed $0 < \theta < 1$, let $\lambda = \theta|m_2(\bar{s})|(2 - \bar{s})^k(R^\gamma\nu)^{-1}$. Set $r = |x - y|$ and

$$v^+(x) = h(y) + \varepsilon + dr^\beta, \text{ in } \bar{\Omega},$$

where $\beta = 2 - \bar{s}$ and

$$d \geq \max \left(\frac{2\bar{m} - h(y) - \varepsilon}{\eta^\beta}, \frac{2\bar{m}\theta^{1/k}}{(1 - \theta^{1/k})R^\beta} \right).$$

It is easy to see that $v^+(y) = h(y) + \varepsilon$, $v^+ \geq 2\bar{m}$, on $\bar{\Omega} \setminus B_\eta(y)$, and by (8.1) $v^+ \geq h$ on $\partial\Omega$. Set $c = h(y) + \varepsilon$, and observing that $\gamma - k\beta > 0$, we calculate, using the value of λ ,

$$\begin{aligned} H(Dv^+, D^2v^+) + \lambda a(x)(v^+)^k &\leq \lambda\nu(c + dR^\beta)^k - \frac{\beta^k d^k}{R^{\gamma-k\beta}}|m_2(\bar{s})| \\ &= (c + dR^\beta)^k \left(\lambda\nu - \frac{\beta^k|m_2(\bar{s})|}{R^\gamma} \left(\frac{dR^\beta}{c + dR^\beta} \right)^k \right) \leq (c + dR^\beta)^k \left(\lambda\nu - \frac{\beta^k|m_2(\bar{s})|}{R^\gamma} \left(\frac{dR^\beta}{2\bar{m} + dR^\beta} \right)^k \right) \\ &\leq (c + dR^\beta)^k \left(\lambda\nu - \frac{|m_2(\bar{s})|\beta^k\theta}{R^\gamma} \right) = 0, \end{aligned}$$

where we have used $t/(1+t)$, $t > 0$, is increasing in t and $dR^\beta \geq 2\bar{m}\theta^{1/k}(1 - \theta^{1/k})^{-1}$. Thus, v^+ is a super-solution. Lemma 5.3 and Remark 2.1 imply existence of a solution u of (1.21).

Case (b): Let (1.10) (ii) hold i.e, Ω satisfies a uniform outer ball condition. Call 2ρ as the optimal radius. Let $z \in \mathbb{R}^n$ be such that $B_{2\rho}(z) \subset \mathbb{R}^n \setminus \Omega$ and $y \in \partial B_{2\rho}(z) \cap \partial\Omega$. Choose $\eta < \rho$.

Using (2.7) fix $\beta > \bar{s} - 2 \geq 0$ and $s = \beta + 2$. Taking $v^\mp = c \pm dr^{-\beta}$,

$$(8.4) \quad H(Dv^+, D^2v^+) \geq \frac{(d\beta)^k|m_2(s)|}{r^{k\beta+\gamma}} > 0, \text{ and } H(Dv^-, D^2v^-) \leq -\frac{(d\beta)^k|m_2(s)|}{r^{k\beta+\gamma}} < 0.$$

We construct a sub-solution as follows. Let p , on the segment \overline{yz} , be such that $|y - p| = \eta/4$. Clearly, $\Omega \cap (B_{\eta/2}(p) \setminus \overline{B}_{\eta/4}(p))$ is a non-empty open set. Set $r = |x - p|$.

Assume that $h(y) > m$ and $m - 2\varepsilon > 0$. We take $v^-(x) = c + dr^{-\beta}$ in $B_{\eta/2}(p)$, where

$$c = h(y) - \varepsilon - \frac{4^\beta d}{\eta^\beta}, \text{ and } d = \eta^\beta \left(\frac{h(y) - \varepsilon - (m/2)}{4^\beta - 2^\beta} \right) > 0.$$

Thus, $v^-(y) = v^-(\eta/4) = h(y) - \varepsilon$, $v^-(x) = m/2$, on $\partial B_{\eta/2}(p)$, and $m/2 \leq v^- \leq h(y) - \varepsilon$. Extend $v^- = m/2$ in $\bar{\Omega} \setminus B_{\eta/2}(p)$.

By (8.4), $H(Dv^-, D^2v^-) + \lambda a(x)(v^-)^k \geq 0$, in $B_{\eta/2}(p) \cap \Omega$. Since $B_{\eta/2}(p) \subset B_\eta(y)$, (8.1) implies $v^- \leq h$ on $\partial\Omega$. The proof that v^- is a sub-solution in Ω is similar to that in Case (a).

We construct super solutions as follows. Take $0 < \theta < 1$ and

$$(8.5) \quad \lambda = \theta \frac{|m_2(s)|\beta^k}{\nu R^\gamma} \left(\frac{\rho}{R} \right)^{k\beta}.$$

Recalling the outer ball condition, select q on the segment \overline{yz} such that $|q - y| = \rho$ and set $r = |x - q|$. There is a $\bar{\rho} > \rho$ such that $B_{\bar{\rho}}(q) \cap \Omega \subset B_{\eta}(y) \cap \Omega$ (2ρ is the optimal radius). Set $v^+(x) = c - dr^{-\beta}$, where

$$c = h(y) + \varepsilon + \frac{d}{\rho^\beta}, \quad \text{and} \quad d \geq \max \left\{ \frac{(\rho\bar{\rho})^\beta(2\bar{m} - h(y) - \varepsilon)}{\bar{\rho}^\beta - \rho^\beta}, \frac{2\bar{m}\theta^{1/k}\rho^\beta}{1 - \theta^{1/k}} \right\}.$$

Clearly, $v^+(y) = v^+(\rho) = h(y) + \varepsilon$, $v^+ \geq 2\bar{m}$, in $\bar{\Omega} \setminus B_{\bar{\rho}}(p)$, and (8.1) implies $v^+ \geq h$ on $\partial\Omega$. Next, using (8.4), (8.5) and $d\rho^{-\beta} \geq 2\bar{m}\theta^{1/k}(1 - \theta^{1/k})^{-1}$, we calculate, in $\Omega \setminus \bar{B}_\rho(q)$,

$$\begin{aligned} H(Dv^+, D^2v^+) + \lambda a(x)(v^+)^k &\leq \lambda\nu(c - dr^{-\beta})^k - \frac{(d\beta)^k |m_2(s)|}{R^{k\beta+\gamma}} \\ &\leq \theta(2\bar{m} + d\rho^{-\beta})^k \frac{|m_2(s)|\beta^k}{R^\gamma} \left(\frac{\rho}{R}\right)^{k\beta} - \frac{\beta^k d^\beta |m_2(s)|}{R^{k\beta+\gamma}} \\ &\leq (2\bar{m} + d\rho^{-\beta})^k \frac{|m_2(s)|\beta^k}{R^\gamma} \left(\frac{\rho}{R}\right)^{k\beta} \left\{ \theta - \left(\frac{d\rho^{-\beta}}{2\bar{m} + d\rho^{-\beta}}\right)^k \right\} \leq 0. \end{aligned}$$

Thus, v^+ is a super-solution and Lemma 5.3 and Remark 2.1 imply existence. \square

Remark 8.1. The proof of Theorem 1.4 shows that, unlike the super-solutions, the constructions of the sub-solutions in Cases (a) and (b) are independent of λ for $\lambda \geq 0$. Also, the upper bounds for λ in the two cases do not depend on the boundary data h . \square

We show a domain monotonicity property of λ_Ω (see (7.5)) i.e, if $\Omega' \subset \Omega$ then $\lambda_\Omega \leq \lambda_{\Omega'}$. We use this in proving Theorem 1.5. This is shown for any subdomain if (1.10) (i) holds and for any subdomain that satisfies a uniform outer ball condition if (1.10) (ii) holds, see (1.20).

Lemma 8.2. *Let $\Omega' \subset \Omega$ be a sub-domain. Suppose that $\lambda > 0$, $\delta > 0$ and $a \in C(\Omega) \cap L^\infty(\Omega)$, $a > 0$. Assume that for some $0 < \lambda < \infty$ the problem*

$$(8.6) \quad H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \text{ in } \Omega, \text{ and } u = \delta \text{ on } \partial\Omega,$$

has a positive solution $u \in C(\bar{\Omega})$. Then the problem $H(Dv, D^2v) + \lambda a(x)|v|^{k-1}v = 0$, in Ω' , and $v = \delta$ on $\partial\Omega'$, has a positive solution $v \in C(\bar{\Omega}')$.

Proof. By Theorem 1.4 and (7.5), $\lambda_{\Omega'} > 0$. Assume that $\lambda_{\Omega'} < \infty$ otherwise the lemma holds.

Suppose that (8.6) has a solution for $\lambda \geq \lambda_{\Omega'}$. By Theorem 7.4 (iv) and (7.5), for any $0 < \hat{\lambda} < \lambda_{\Omega'}$, there is a solution $v_{\hat{\lambda}} \in C(\bar{\Omega}')$, $v_{\hat{\lambda}} > 0$, of

$$H(Dv_{\hat{\lambda}}, D^2v_{\hat{\lambda}}) + \hat{\lambda}a(x)v_{\hat{\lambda}}^k = 0, \text{ in } \Omega', \text{ and } v_{\hat{\lambda}} = \delta \text{ on } \partial\Omega'.$$

By Theorem 7.4 (i), $u \geq \delta$ on $\partial\Omega'$. Applying Theorem 1.1 in Ω' , $v_{\hat{\lambda}} \leq u$ for every $\hat{\lambda} \in (0, \lambda_{\Omega'})$. Since u is bounded, this contradicts Theorem 7.4(iii). The claim holds and $\lambda < \lambda_{\Omega'}$. \square

Remark 8.3. Take $h \in C(\partial\Omega)$ with $\inf_{\partial\Omega} h > 0$. For some $\lambda > 0$, let $u > 0$ solve $H(Du, D^2u) + \lambda a(x)u^k = 0$, in Ω , and $u = h$ on $\partial\Omega$. Recall λ_Ω from (7.5) and let $\lambda \geq \lambda_\Omega$. For every $t < \lambda_\Omega$, let $v_t > 0$ solve $H(Dv_t, D^2v_t) + ta(x)v_t^k = 0$, in Ω , and $v_t = \inf_{\partial\Omega} h$ on $\partial\Omega$. By Theorem 1.1, $v_t \leq u$, $\forall t$. This contradicts Theorem 7.4(iii). Thus, $H(Dw, D^2w) + \lambda a(x)w^k = 0$, in Ω and $w = \delta > 0$, on $\partial\Omega$ has a solution. \square

We now prove Theorem 1.5 and show that λ_Ω in (7.5) is independent of the data $h \in C(\partial\Omega)$, $h > 0$. See Remark 8.1.

Proof of Theorem 1.5. Let $\lambda > 0$ such that there is a solution $u \in C(\overline{\Omega})$, $u > 0$, of $H(Du, D^2u) + \lambda a(x)u^k = 0$, in Ω , and $u = \delta$ on $\partial\Omega$. We show that

$$(8.7) \quad H(Dv, D^2v) + \lambda a(x)v^k = 0, \text{ in } \Omega, \text{ and } v = h \text{ on } \partial\Omega,$$

can be solved for any boundary data $h \in C(\partial\Omega)$ with $\inf_{\partial\Omega} h > 0$.

By Remark 8.1, the sub-solutions in Theorem 1.4 can be utilized here. Thus, our effort is to construct super-solutions to (8.7) for a given h . Set

$$\nu = \sup_{x \in \Omega} a(x), \quad m = \inf_{\partial\Omega} h, \quad \bar{m} = \sup_{\partial\Omega} h, \text{ and } R = \text{diam}(\Omega); \text{ assume that } m > 0 \text{ and } \nu < \infty.$$

Fix $y \in \partial\Omega$. Let $\varepsilon > 0$ such that $\varepsilon < \bar{m}/2$. Let $\eta_0 > 0$ be such that for $0 < \eta \leq \eta_0$,

$$(8.8) \quad h(y) - \varepsilon \leq h(x) \leq h(y) + \varepsilon, \quad \forall x \in \overline{B_\eta(y)} \cap \partial\Omega.$$

Let u solve (8.7) with $\delta = 2\bar{m}$ and call $M = \sup_\Omega u$. Set $\Omega_\eta = \Omega \setminus \overline{B_\eta(y)}$. By Lemma 8.2, there is a unique solution $\phi > 0$ of

$$(8.9) \quad H(D\phi, D^2\phi) + \lambda a(x)\phi^k = 0, \text{ in } \Omega_\eta, \text{ and } \phi = 2\bar{m} \text{ on } \partial\Omega_\eta.$$

By Lemma 6.1, $u \geq 2\bar{m}$ on $\partial\Omega_\eta$. Using Lemma 5.3, (8.9), Lemmas 4.2 and 6.1, $2\bar{m} < \phi \leq u \leq M$, for any $\eta > 0$, small.

Case (a): Suppose that (1.10) (i) holds and Ω is any domain. Set $r = |x - y|$ and

$$(8.10) \quad \bar{\phi}(x) = h(y) + \varepsilon + (2\bar{m} - h(y) - \varepsilon) \frac{r^\beta}{\eta^\beta}, \text{ in } \overline{B_\eta(y)} \cap \Omega,$$

where $\beta = 2 - \bar{s}$, see (1.10). Recalling (8.9) and (8.10), we define

$$(8.11) \quad w(x) = \begin{cases} \phi(x), & \forall x \in \Omega_\eta, \\ \bar{\phi}(x), & \forall x \in \overline{B_\eta(y)} \cap \Omega. \end{cases}$$

Note that $w \in C(\overline{\Omega})$, $w(y) = h(y) + \varepsilon$, $w = 2\bar{m}$, on $\partial B_\eta(y)$, and $w \geq h$ on $\partial\Omega$. Our goal is to choose $0 < \eta \leq \eta_0$ (see (8.8)) such that w is a super-solution in Ω . By (2.6)(also see (8.2)) there

is an $\eta_1 \in (0, \eta_0]$ such that in $B_\eta(y) \cap \Omega$, $\forall \eta \in (0, \eta_1]$, we have

$$\begin{aligned} H(D\bar{\phi}, D^2\bar{\phi}) + \lambda a(x)\bar{\phi}^k &\leq \lambda \nu \bar{\phi}^k - \left(\frac{2\bar{m} - h(y) - \varepsilon}{\eta^\beta} \right)^k \frac{\beta^k |m_2(\bar{s})|}{r^{\gamma-k\beta}} \\ (8.12) \qquad \qquad \qquad &\leq \lambda \nu (2\bar{m})^k - (2\bar{m} - h(y) - \varepsilon)^k \frac{\beta^k |m_2(\bar{s})|}{\eta^\gamma} \leq 0. \end{aligned}$$

Here we have used $0 < r < \eta$ and $\gamma - k\beta > 0$. We show that w is a super-solution on $\partial B_\eta(y) \cap \Omega$ implying that w is a super-solution in Ω . Our idea is to choose a value of η so that $d\bar{\phi}/dr(\eta)$ exceeds the radial rate of increase of ϕ on $r = \eta$.

We now estimate ϕ (in Ω_η) on $\partial B_\eta(y)$, from above. Note $2\bar{m} < \bar{\phi} \leq M$ (see (8.9)). Set

$$(8.13) \qquad \theta = \left(1 + \frac{2(2M - 2\bar{m})}{2\bar{m} - h(y) - \varepsilon} \right)^{1/\beta},$$

where $\beta = 2 - \bar{s}$, see (1.10)(i). In $\eta \leq r \leq \theta\eta$, set

$$\psi(x) = 2\bar{m} + d(r^\beta - \eta^\beta), \quad \text{where } d = \frac{2M - 2\bar{m}}{\eta^\beta(\theta^\beta - 1)}.$$

Clearly, $\psi = 2\bar{m}$, on $r = \eta$, $\psi = 2M$, on $r = \theta\eta$, and $2\bar{m} \leq \psi \leq 2M$. Using (8.13),

$$(8.14) \qquad d = \frac{2M - 2\bar{m}}{\eta^\beta(\theta^\beta - 1)} = \frac{2\bar{m} - h(y) - \varepsilon}{2\eta^\beta}.$$

We choose $\eta_2 \in (0, \eta_1]$ so that ψ is a super-solution in $\eta < r < \theta\eta$, for any $\eta \in (0, \eta_2]$. This would then imply by Lemma 5.3, that $\phi \leq \psi$, in $\eta < r < \theta\eta$. Employing $\gamma - k\beta > 0$, (2.6) and (8.14)

$$\begin{aligned} H(D\psi, D^2\psi) + \lambda a(x)\psi^k &\leq \lambda \nu (2M)^k - \frac{(d\beta)^k |m_2(\bar{s})|}{r^{\gamma-k\beta}} \leq \lambda \nu (2M)^k - \frac{(d\eta^\beta \beta)^k |m_2(\bar{s})|}{\theta^\gamma \eta^\gamma} \\ &= \lambda \nu (2M)^k - \left(\frac{2M - 2\bar{m}}{\theta^\beta - 1} \right)^k \frac{\beta^k |m_2(\bar{s})|}{\theta^\gamma \eta^\gamma} \leq 0, \end{aligned}$$

if $0 < \eta \leq \eta_2 \leq \eta_1 \leq \eta_0$ are small enough. Thus, (8.8) and (8.12) hold and this gives us the desired ϕ , $\bar{\phi}$ and the upper bound ψ .

We show that w is a super-solution. Let $\phi \in C^2$ be such that $w - \phi$ has a minimum at some $p \in \partial B_\eta(y) \cap \Omega$. Then (i) $\varphi(x) - \varphi(p) \leq w(x) - w(p) = \bar{\phi}(x) - \bar{\phi}(p) \leq 0$, $\forall x \in B_\eta(y) \cap \Omega$, and (ii) $\varphi(x) - \varphi(p) \leq w(x) - w(p) \leq \psi(x) - \psi(p)$, $\forall x \in \Omega_\eta$. Using (8.10) and (8.14), these yield,

$$\frac{\partial \varphi}{\partial r}(p) \geq \bar{\phi}'(\eta-) = \beta \left(\frac{2\bar{m} - h(y) - \varepsilon}{\eta} \right) \quad \text{and} \quad \frac{\partial \varphi}{\partial r}(p) \leq \psi'(\eta+) = \beta \left(\frac{2\bar{m} - h(y) - \varepsilon}{2\eta} \right).$$

This is a contradiction and w is a super-solution in Ω . Lemma 5.3 and Remark 2.1 imply the existence of a solution of (1.22).

Case (b): Assume that (1.10)(ii) holds and Ω satisfies a uniform outer ball condition. Fix $y \in \partial\Omega$ and $\varepsilon > 0$, small. Let $\rho > 0$ and $z \in \mathbb{R}^n$ be such that $B_\rho(z) \subset \mathbb{R}^n \setminus \Omega$ and $y \in \partial B_\rho(z) \cap \partial\Omega$. Set $r = |x - z|$. We modify (8.8) as follows. We choose $\eta_0 > 0$, small, such that for $0 < \eta \leq \eta_0$,

$$h(y) - \varepsilon \leq h(x) \leq h(y) + \varepsilon, \quad \forall x \text{ such that } \rho \leq |x - z| \leq \rho + \eta.$$

The ideas are similar to those in Case (a). Define $\Omega_\eta = \Omega \setminus \overline{B}_{\rho+\eta}(z)$ and note that Ω_η satisfies a uniform outer ball condition. For $\eta \in (0, \eta_0]$, to be determined, there is a solution $\phi > 0$ of

$$(8.15) \quad H(D\phi, D^2\phi) + \lambda a(x)\phi^k = 0, \quad \text{in } \Omega_\eta, \quad \text{and } \phi = 2\bar{m} \text{ on } \partial\Omega_\eta,$$

by our hypothesis and Lemma 8.2. As noted in (8.9), $2\bar{m} < \phi \leq M$.

Call $A_\eta = \{x \in \Omega : \rho < |x - z| < \rho + \eta\}$. We fix $\beta > \bar{s} - 2$ and $s = \beta + 2$, and define

$$(8.16) \quad \bar{\phi}(x) = h(y) + \varepsilon + \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{-\beta} - (\rho + \eta)^{-\beta}} \right) \left(\frac{1}{\rho^\beta} - \frac{1}{r^\beta} \right), \quad \rho \leq r \leq \rho + \eta.$$

It is clear that $\bar{\phi}(y) = h(y) + \varepsilon$, $\bar{\phi} = 2\bar{m}$ on $r = \rho + \eta$, and $h(y) + \varepsilon \leq \bar{\phi} \leq 2\bar{m}$. Using (8.15) and (8.16), we define

$$(8.17) \quad w(x) = \begin{cases} \phi(x), & \forall x \in \Omega_\eta, \\ \bar{\phi}(x), & \forall x \in A_\eta. \end{cases}$$

As done in Case (a), we will select $\eta > 0$ such that w is a super-solution in Ω . Clearly, $w(y) = h(y) + \varepsilon$ and $w \geq h$ on $\partial\Omega$. Using (2.7) (see (8.4)) in A_η , i.e., $\rho \leq r \leq \rho + \eta$,

$$\begin{aligned} H(D\bar{\phi}, D^2\bar{\phi}) + \lambda a(x)\bar{\phi}^k &\leq \lambda\nu(2\bar{m})^k - \frac{\beta^k |m_2(s)|}{r^{k\beta+\gamma}} \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{-\beta} - (\rho + \eta)^{-\beta}} \right)^k \\ &\leq \lambda\nu(2\bar{m})^k - \frac{\beta^k |m_2(s)|}{(\rho + \eta)^{k\beta+\gamma}} \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{-\beta} - (\rho + \eta)^{-\beta}} \right)^k. \end{aligned}$$

Thus, $\bar{\phi}$ is a super-solution in A_η , if $\eta \in (0, \eta_1]$, for some $0 < \eta_1 \leq \eta_0$, small.

Next we choose $0 < \eta_2 \leq \eta_1$ so that, for $0 < \eta \leq \eta_2$, the quantity

$$(8.18) \quad K = \left(\frac{\rho^\beta}{(\rho + \eta)^\beta - \rho^\beta} \right) \left(\frac{2\bar{m} - h(y) - \varepsilon}{2(2M - 2\bar{m})} \right) > 1.$$

We calculate an upper bound for ϕ in $\Omega_\eta \cap B_{\rho+\theta\eta}(z)$, where $\theta > 1$ is to be determined. We take

$$(8.19) \quad \psi(x) = 2\bar{m} + \left(\frac{2M - 2\bar{m}}{(\rho + \eta)^{-\beta} - (\rho + \theta\eta)^{-\beta}} \right) \left(\frac{1}{(\rho + \eta)^\beta} - \frac{1}{r^\beta} \right), \quad \rho + \eta \leq r \leq \rho + \theta\eta.$$

Then $\psi = 2\bar{m}$, on $r = \rho + \eta$, and $\psi = 2M$, on $r = \rho + \theta\eta$. We calculate θ by requiring that $\bar{\phi}'(\rho + \eta) > \psi'(\rho + \eta)$, in particular, we impose that

$$\left(\frac{2M - 2\bar{m}}{(\rho + \eta)^{-\beta} - (\rho + \theta\eta)^{-\beta}} \right) = \frac{1}{2} \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{-\beta} - (\rho + \eta)^{-\beta}} \right),$$

see (8.16) and (8.17). Rearranging and recalling (8.18),

$$\frac{(\rho + \theta\eta)^\beta}{(\rho + \theta\eta)^\beta - (\rho + \eta)^\beta} = \left(\frac{\rho^\beta}{(\rho + \eta)^\beta - \rho^\beta} \right) \left(\frac{2\bar{m} - h(y) - \varepsilon}{2(2M - 2\bar{m})} \right) = K.$$

By (8.15), $(\rho + \theta\eta)^\beta = K(\rho + \eta)^\beta / (K - 1)$. Clearly, $\theta = \theta(\eta) > 1$ if $0 < \eta \leq \eta_2$.

We now show that ψ is super-solution in $\rho + \eta < r < \rho + \theta\eta$, if η is small enough. Using (2.7)

$$H(D\psi, D^2\psi) + \lambda a(x)\psi^k \leq \lambda\nu(2M)^k - \left(\frac{2M - 2\bar{m}}{(\rho + \eta)^{-\beta} - (\rho + \theta\eta)^{-\beta}} \right)^k \frac{\beta^k |m_2(s)|}{(\rho + \theta\eta)^{k\beta+\gamma}}$$

Thus, ψ is a super-solution if $\eta \in (0, \eta_3]$, where $\eta_3 \in (0, \eta_2]$, small. This determines θ . As done in Case (a), w is a super-solution in Ω . Lemma 5.3 and Remark 2.1 imply existence. \square

9. BOUNDEDNESS OF λ_Ω : PROOF OF THEOREM 1.6

In this section, we show that λ_Ω is bounded, see Theorem 7.4. In the first part of the proof we assume that (1.10)(i) holds and impose conditions A , B and C . In the second part we address the case (1.10) (ii) and conditions A , B , C and D (radial symmetry) are assumed.

Let $a(x) \in C(\Omega) \cap L^\infty(\Omega)$, $\inf_\Omega a > 0$, and $\delta > 0$; we consider the problem

$$(9.1) \quad H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \text{ in } \Omega \text{ and } u = \delta \text{ on } \partial\Omega.$$

We recall the definition of λ_Ω :

$$(9.2) \quad \lambda_\Omega = \sup\{\lambda : (9.1) \text{ has a positive solution}\}.$$

By Theorem 7.4, if $0 < \lambda < \lambda_\Omega$ in (9.1) then $u \in C(\overline{\Omega})$, $u > \delta$, and u is unique.

Remark 9.1. Let $b \in C(\Omega)$ with $a(x) \leq b(x)$, $\forall x \in \Omega$. For some $\lambda > 0$, let $v \in C(\overline{\Omega})$ solve

$$H(Dv, D^2v) + \lambda b(x)|v|^{k-1}v = 0, \text{ in } \Omega, v > 0 \text{ and } v = \delta \text{ on } \partial\Omega.$$

Then (9.1) has a solution $u > 0$ for λ . Note that v is super-solution of (9.1), i.e., $H(Dv, D^2v) + \lambda a(x)|v|^{k-1}v \leq 0$, in Ω , and $v = \delta$ on $\partial\Omega$. Also, $w = \delta$ is a sub-solution of (9.1). By Lemma 5.3 and Remark 2.1, there is solution $w \leq u \leq v$ of (9.1) for λ . Let $\lambda_\Omega(c)$ be the bound in (9.2) for the weight function $c(x)$ then

$$(9.3) \quad \lambda_\Omega(b) \leq \lambda_\Omega(a), \text{ for } a(x) \leq b(x), \forall x \in \Omega. \quad \square$$

We now prove Theorem 1.6.

Proof of Theorem 1.6. Set $\mu = \inf_{x \in \Omega} a(x)$ and assume that $\mu > 0$. Let $B_R(y)$, where $y \in \Omega$, denote an in-ball of Ω . Call $B = B_R(y)$. By Lemma 8.2, if $0 < \lambda < \lambda_\Omega$ then $0 < \lambda < \lambda_B$.

Consider the problem

$$(9.4) \quad H(Dv, D^2v) + \lambda \mu v^k = 0, \text{ in } B, \text{ and } v = \delta \text{ on } \partial B.$$

By (9.3), if (9.1) has a solution on B for some $\lambda > 0$ then (9.4) has a positive solution since $\lambda_B(\mu) \geq \lambda_B(a)$. Thus, if we show that $\lambda_B(\mu) < \infty$ then $\lambda_\Omega(a) < \infty$. We set

$$\lambda = \lambda_\mu \text{ and } \lambda_R = \mu \lambda_B(\mu).$$

We assume that $\lambda_R = \infty$ and derive a contradiction.

Let $\lambda_1 > 0$ and recall that $k = k_1 + k_2$ and $\gamma = k_1 + 2k_2$, see (1.4). Set $\lambda_\ell = \ell^\gamma \lambda_1$, $\ell = 1, 2, \dots$. By Theorem 7.4, there is an $u_\ell \in C(\overline{B})$, $u_\ell > 0$, that solves

$$(9.5) \quad H(Du_\ell, D^2u_\ell) + \lambda_\ell u_\ell^k = 0 \text{ in } B, \text{ and } u_\ell = \delta_\ell \text{ on } \partial B, \forall \ell = 1, 2, \dots,$$

where $\delta_\ell > 0$ is so chosen that $u_\ell(y) = 1$.

For each ℓ , (9.4) has a unique solution $v_\ell > 0$ in B with $v_\ell = 1$ on ∂B . Thus by Lemma 5.3, $u_\ell/v_\ell \leq \delta_\ell$ and $v_\ell/u_\ell \leq 1/\delta_\ell$ implying $u_\ell = \delta_\ell v_\ell \neq 0$. This shows that $\delta_\ell > 0$.

Step 1: We show that δ_ℓ decreases to zero. Recall from (1.9) that $\alpha = \gamma/k$. For $\forall \ell = 1, 2, \dots$, set $\tau_\ell = (\lambda_\ell/\lambda_{\ell+1})^{1/k}$, i.e, $\tau_\ell = (\ell/(\ell+1))^\alpha$.

Applying Remark 5.5(ii) and Lemma 5.6, $u_\ell(x)[u_{\ell+1}(x)]^{-\tau_\ell} \leq \delta_\ell(\delta_{\ell+1})^{-\tau_\ell}$. Taking $x = y$, we have $\delta_{\ell+1}^{\tau_\ell} \leq \delta_\ell$ implying $(\delta_{\ell+1})^{(\ell+1)^{-\alpha}} \leq (\delta_\ell)^{\ell^{-\alpha}}$. Iterating

$$(9.6) \quad 0 < \delta_{\ell+1} \leq (\delta_\ell)^{1/\tau_\ell} \leq (\delta_{\ell-1})^{1/(\tau_\ell \tau_{\ell-1})} \leq \dots \leq (\delta_1)^{(\ell+1)^\alpha}.$$

Since $\delta_1 < 1$, the claim follows by letting $\ell \rightarrow \infty$.

Step 2: We employ scaling. For $\forall \ell = 1, 2, \dots$, call $R_\ell = \ell R$ and $v_\ell(z) = u_\ell(x)$, $\forall z \in B_{R_\ell}(y)$, where $z = y + \ell(x - y)$. Then $H(Du_\ell, D^2u_\ell) = \ell^\gamma H(Dv_\ell, D^2v_\ell)$, $\lambda_\ell = \lambda_1 \ell^\gamma$ and (9.5) imply that

$$(9.7) \quad H(Dv_\ell, D^2v_\ell) + \lambda_1 v_\ell^k = 0, \text{ in } B_{R_\ell}(y), \text{ and } v_\ell(R_\ell) = \delta_\ell.$$

Moreover, $v_\ell(y) = u_\ell(y) = 1$. Call $B_\ell = B_{R_\ell}(y)$.

We address (1.10) (i) and (ii) separately. Set $r = |x - y|$ and recall (2.6) and (2.7).

Case (a): Suppose that (1.10)(i) holds; fix $\beta = 2 - \bar{s}$ where $0 < \beta < 1$. For m , large, take

$$w_m(x) = w_m(r) = \frac{1}{2} \left(1 - (1 - \delta_m) \left(\frac{r}{R_m} \right)^\beta \right), \quad 0 < r \leq R_m.$$

Note that $w_m(y) = 1/2$ and $w_m(R_m) = \delta_m/2$. Moreover, $w > 0$ and the calculation in (2.6) shows that $H(Dw_m, D^2w_m) + \lambda_1 w_m^k \geq 0$, in $B_{R_m}(y) \setminus \{y\}$. Applying Lemma 5.3 to w_m and v_m in $B_m \setminus \{y\}$, we see that

$$(9.8) \quad w_m(x) \leq v_m(x)/2, \quad \forall x \in B_m(y).$$

Step a(i) : Let $1 < \ell \leq m$. Applying Lemma 5.3, we see that

$$\frac{v_\ell(x)}{v_m(x)} \leq \frac{\delta_\ell}{\inf_{\{r=R_\ell\}} v_m} \quad \text{and} \quad \frac{v_m(x)}{v_\ell(x)} \leq \frac{\sup_{\{r=R_\ell\}} v_m}{\delta_\ell}, \quad \forall x \in B_\ell.$$

Taking $x = y$ in the above inequality, we get $\inf_{\{r=R_\ell\}} v_m \leq \delta_\ell \leq \sup_{\{r=R_\ell\}} v_m$. Using (9.8), we obtain that for each $\ell \leq m$,

$$w_m(R_\ell) \leq \delta_\ell/2.$$

Step a(ii) : Taking $m = 2\ell$, recalling that $R_{2\ell} = 2R_\ell$, using Step a(i) and Step 2, we obtain that

$$\frac{1}{2} \left(1 - \frac{1 - \delta_{2\ell}}{2^\beta} \right) = w_{2\ell}(R_\ell) \leq \frac{\inf_{\{r=R_\ell\}} v_{2\ell}}{2} \leq \frac{\delta_\ell}{2}.$$

Letting $\ell \rightarrow \infty$, we obtain a contradiction to (9.6).

Case (b): Assume that (1.10)(ii) holds. Fix $\beta = s - 2$, where $s > \bar{s}$. We impose conditions A , B , C and D . See Remark 9.2 in this context. By condition D , H is invariant under rotations.

Step b(i): Using condition D, Lemma 5.3 implies that v_ℓ is radial, see Step 2. By Lemma 6.1, $v_\ell(r) \geq v_\ell(\rho)$, for $0 \leq r \leq \rho \leq R_\ell$. Thus, $v_\ell(r)$ is non-increasing and $\sup v_\ell = v_\ell(y) = 1$.

Step b(ii): Let $1 \leq \ell \leq m$, applying Lemma 5.3 in B_ℓ we obtain that

$$1 = \frac{v_m(o)}{v_\ell(o)} \leq \frac{v_m(R_\ell)}{\delta_\ell} \quad \text{and} \quad 1 = \frac{v_\ell(o)}{v_m(o)} \leq \frac{\delta_\ell}{v_m(R_\ell)}.$$

Clearly, $v_m(R_\ell) = \delta_\ell$. Lemma 5.3 implies that $v_m = v_\ell$, in B_ℓ . Thus, v_m extends v_ℓ to B_m .

Step b(iii): We claim that the decay estimate in Step 1 can not hold, leading to a contradiction and thus proving that $\lambda_\Omega < \infty$. We proceed as follows. By Step b(i), there is a $0 < \rho < R$ such that for any ℓ , $v_\ell > 1/2$, in $B_\rho(y)$. Consider the function

$$\omega(x) = \frac{1}{2} \left(\delta_{2\ell} + (1 - \delta_{2\ell}) \frac{r^{-\beta} - R_{2\ell}^{-\beta}}{\rho^{-\beta} - R_{2\ell}^{-\beta}} \right), \quad \forall \rho \leq r \leq R_{2\ell}.$$

Using (2.7), $H(D\omega, D^2\omega) + \lambda\nu\omega^k \geq 0$, in $\rho < r < R_{2\ell}$. Since $\omega(\rho) = 1/2$ and $\omega(R_{2\ell}) = \delta_{2\ell}/2$, applying Lemma 5.3 in $B_{R_{2\ell}} \setminus B_\rho(y)$ we see that $\omega(x) \leq v_{2\ell}(x)$, in $\rho \leq r \leq R_{2\ell}$. Recall Steps 1, 2 and Step b(ii), and take $r = R_\ell$ to find

$$\frac{C}{\ell^\beta} \leq \omega(R_\ell) \leq v_{2\ell}(R_\ell) = \delta_\ell \leq (\delta_1)^{\ell^\alpha},$$

where $C > 0$, depends on β, ρ and R . Letting $\ell \rightarrow \infty$, we obtain a contradiction. \square

Remark 9.2. In Case (b), condition D is not required if non-negative super-solutions w i.e, $H(Dw, D^2w) \leq 0$, satisfy a Harnack inequality i.e, $\inf_{B_s(z)} w \geq C \sup_{B_s(z)} w$, where C is a universal constant and $B_{4s}(z) \subset B$. In Step b(iii), $\forall \ell$, $v_\ell(x) \geq C$, $\forall x \in B_\rho(y)$, with $0 < \rho \leq R/8$. Apply Steps a(i), (ii) and b(iii) to get a proof. Also, the proof works if w satisfies a modulus of continuity depending on $\sup w$. \square

10. EXISTENCE OF A POSITIVE FIRST EIGENFUNCTION

This section has two sub-sections. In Sub-section I, we show the existence of a positive eigenfunction on a general domain when (1.10)(i) holds and conditions A , B and C apply. In Sub-section II, we discuss (1.10)(ii), impose conditions A , B , C and D and take Ω to be a ball.

Sub-section I: Let $\Omega \subset \mathbb{R}^n$ be any bounded domain and assume that (1.10) (i) holds. Fix $\beta = 2 - \bar{s}$, $0 < \beta < 1$, recall (2.6) and take $a \in C(\Omega) \cap L^\infty(\Omega)$, $\inf_\Omega a > 0$. See [1, 5, 6, 11].

Lemma 10.1. (*The Harnack inequality and Hölder Continuity*) Let $w \in lsc(\Omega) \cap L^\infty(\Omega)$, $w \geq 0$, solve $H(Dw, D^2w) + \lambda a(x)|w|^{k-1}w \leq 0$, in Ω . For any $y \in \Omega$ and $R > 0$ such that $B_{4R}(y) \subset \Omega$, we have a universal constant $C > 0$ such that

$$\sup_{B_R(y)} w \leq C \inf_{B_R(y)} w, \quad \text{and} \quad |w(x) - w(z)| \leq (3R)^{-\beta} \left(\sup_{B_R(y)} w \right) |x - z|^\beta, \quad \forall x, z \in B_R(y).$$

Proof. Let $w(y) > 0$, for some $y \in \Omega$, and $B_{4R}(y) \subset \Omega$. Set $A = B_{4R}(y) \setminus \{y\}$, $r = |x - y|$ and

$$\psi(x) = w(y) \left(1 - r^\beta (4R)^{-\beta}\right), \text{ in } B_{4R}(y).$$

Thus, $\psi(y) = w(y)$, $\psi = 0$, on $\partial B_{4R}(y)$ and by (2.6), $H(D\psi, D^2\psi) > 0$ in A . Clearly, on ∂A , $\inf(w - \psi) = 0$. If $\inf_A(w - \psi) < 0$ and $p \in A$ is a point of minimum then $H(D\psi(p), D^2\psi(p)) + \lambda a(p)|w(p)|^{k-1}w(p) > 0$, which is a contradiction. Thus, $w(x) \geq \psi(x) > 0$, in $B_{4R}(y)$.

Observing that for any $z \in B_R(y)$, $B_R(y) \subset B_{2R}(z)$ and arguing as above,

$$(10.1) \quad w(x) \geq w(z) \left(1 - |x - z|^\beta (3R)^{-\beta}\right) \text{ for any } x, z \in B_R(y).$$

Since $|x - z| \leq 2R$, the claim holds. To show the Hölder continuity of w , we write (10.1) as

$$w(z) - w(x) \leq w(z)|x - z|^\beta (3R)^{-\beta} \leq (3R)^{-\beta} \left(\sup_{B_R(y)} w\right) |x - z|^\beta.$$

Taking $x \in B_R(y)$ and replacing x by z , we get the claim. \square

Proof of Theorem 1.7(i). Let $\delta > 0$ and λ_ℓ , $\ell = 1, 2, \dots$ be an increasing sequence such that $\lim_{\ell \rightarrow \infty} \lambda_\ell = \lambda_\Omega$. For each ℓ , there is a unique positive $u_\ell \in C(\overline{\Omega})$ such that

$$H(Du_\ell, D^2u_\ell) + \lambda_\ell a(x)u_\ell^k = 0, \text{ in } \Omega, \text{ and } u_\ell = \delta, \text{ on } \partial\Omega.$$

Set $\theta_\ell = \sup_\Omega u_\ell$. By Theorem 7.4, $\lim_{\ell \rightarrow \infty} \theta_\ell = \infty$. Calling $v_\ell = u_\ell/\theta_\ell$, we see that $\sup v_\ell = 1$ and $v_\ell|_{\partial\Omega} \rightarrow 0$. By Lemma 10.1, there is a sub-sequence v_m and a function $v \in C(\Omega)$ such that v_m converges uniformly to v on compact subsets. Thus,

$$v \geq 0, \quad \sup_\Omega v = 1 \quad \text{and} \quad \lim_{m \rightarrow \infty} v_m = \lim_{m \rightarrow \infty} \frac{\delta}{\theta_m} = 0 \quad \text{on } \partial\Omega.$$

(a) We show that $H(Dv, D^2v) + \lambda_\Omega a(x)v^k = 0$ in Ω . Let $\phi \in C^2$ and $p \in \Omega$ be such that $v - \phi$ has a minimum at p . Set $B_\varepsilon = B_\varepsilon(p)$, for $\varepsilon > 0$, small. Let $\hat{\Omega}$ be a compact sub-domain of Ω containing B_ε . Set $d = \text{dist}(p, \partial\hat{\Omega})$, $k = \max(3, 3d^{-4})$ and $\psi = \phi(x) - k|x - p|^4$. Take m , large, so that $\sup_\Omega |v_m - v| \leq \varepsilon^4$ on $\hat{\Omega}$. Then

$$v_m(x) - \psi(x) \geq k|x - p|^4 + (v_m - v)(x) + (v - v_m)(p) + (v_m - \psi)(p).$$

Noting that $v_m - \psi > (v_m - \psi)(p)$, on $\Omega \setminus B_\varepsilon$, $v_m - \psi$ has a minimum at some $p_m \in B_\varepsilon$, and $H(D\psi(p_m), D^2\psi(p_m)) + \lambda_m a(p_m)v_m(p_m)^k \leq 0$. Letting $\varepsilon \rightarrow 0$, $H(D\phi(p), D^2\phi(p)) + \lambda_\Omega a(p)v(p)^k \leq 0$. The proof that v is a sub-solution is similar.

(b) We show that $v \in C(\overline{\Omega})$. Let $y \in \partial\Omega$ and set $r = |x - y|$. Recalling (1.10) (i) and (2.6), take $w_\varepsilon(x) = \varepsilon + (1 - \varepsilon)(r/\rho)^\beta$, in $B_\rho(y)$, where $0 < \varepsilon < 1/4$ is small and ρ is to be determined. Set $\nu = \sup_\Omega a$ and recall that $\gamma - k\beta > 0$. There is a $\rho > 0$, small and independent of ε so that

$$H(Dw_\varepsilon, D^2w_\varepsilon) + \lambda a(x)w_\varepsilon^k \leq \lambda\nu - \frac{(\beta(1 - \varepsilon))^k |m_2(\bar{s})|}{\rho^{k\beta} r^{\gamma - k\beta}} \leq \lambda\nu - \left(\frac{3\beta}{4}\right)^k \frac{|m_2(\bar{s})|}{\rho^\gamma} < 0, \quad 0 < r \leq \rho.$$

Also, $\varepsilon \leq w_\varepsilon \leq 1$, and, for large m , $w_\varepsilon \geq v_m$ on $\partial(\Omega \cap B_\rho(y))$. By Lemma 5.3, $w_\varepsilon \geq v_m$, in $B_\rho(y) \cap \Omega$. Letting $m \rightarrow \infty$, $w_\varepsilon \geq v$, in $\Omega \cap B_\rho(y)$. Clearly, $0 \leq \liminf_{x \rightarrow y} v(x) \leq \limsup_{x \rightarrow y} v(x) \leq w_\varepsilon(y) = \varepsilon$. Since ε is arbitrary, $v(y) = 0$ and $v \in C(\overline{\Omega})$.

(c) Next, we show that $v > 0$ in Ω . Let $p \in \Omega$ be such that $v(p) = 1$. Recall from (b) the bound $v \leq w = (r/\rho)^\beta$, in $B_\rho(y) \cap \Omega$. Clearly, if we take $r = \rho/2$, we have $v < 1$. Thus, p is at least $\rho/2$ away from $\partial\Omega$. We now apply Harnack's inequality in Lemma 10.1 to conclude $v > 0$ in Ω . \square

Sub-section II: Assume that (1.10)(ii) holds. We take Ω to be the ball $B_R(o)$, where $R > 0$, and prove the existence of a positive radial first eigenfunction. Set $\lambda_R = \lambda_{B_R(o)}$, $r = |x|$, $\forall x \in \mathbb{R}^n$ and take $a(x) = 1, \forall x \in \Omega$. We impose conditions A , B , C and D . As observed in Step b(i) of Theorem 1.6 there is a $v \in C(\overline{B_R(o)})$, $v > 0$, radial and non-increasing in r , that solves

$$(10.2) \quad H(Dv, D^2v) + \lambda v^k = 0, \text{ in } B_R(o), v = \delta \text{ on } \partial B_R(o), \text{ and } 0 < \lambda < \lambda_R.$$

Scaling property. For $0 < R_1 < R_2$, set $B_i = B_{R_i}(o)$, $i = 1, 2$. By Lemma 8.2, $\lambda_{R_1} \geq \lambda_{R_2}$. Let u be the solution of (10.2) in B_1 , for some $0 < \lambda < \lambda_{R_1}$. Define $v(y) = u(x)$, $\forall y \in B_2$, where $y = sx$, where $s = R_2/R_1$. As $\gamma = k_1 + 2k_2$, we have $H(Du, D^2u) = s^\gamma H(Dv, D^2v)$ and

$$H(Dv, D^2v) + s^{-\gamma} \lambda v^k = 0, \text{ in } B_2, \text{ and } v = \delta, \text{ on } \partial B_2.$$

By (9.2) and Theorem 7.4 (iv), $\lambda_{R_1} = s^\gamma \lambda_{R_2}$

$$(10.3) \quad \lambda_{R_1} R_1^\gamma = \lambda_{R_2} R_2^\gamma.$$

We now show existence of a first eigenfunction on a ball. Set $r = |x|$, $\forall x \in \mathbb{R}^n$. Call $B = B_R(o)$.

Step 1: Fix $\delta > 0$, $0 < \lambda < \lambda_R$ and let $u = u(r) \in C(\overline{B})$, $u > 0$, be the unique solution of

$$H(Du, D^2u) + \lambda u^k = 0, \text{ in } B, u(R) = \delta \text{ and } u(o) = 1.$$

Set $\hat{R} = (\lambda_R/\lambda)^{1/\gamma} R$. By (10.3), $\lambda_{\hat{R}} = \lambda$, $\hat{R} > R$ and λ is the first eigenvalue of H on $B_{\hat{R}}(o)$.

Step 2: Let $\{\lambda_\ell\}_{\ell=1}^\infty$ be a decreasing sequence such that $\lambda < \lambda_\ell < \lambda_R$ and $\lim_{\ell \rightarrow \infty} \lambda_\ell = \lambda$. Call R_ℓ such that $\lambda_\ell R_\ell^\gamma = \lambda_R R^\gamma$ and $B_\ell = B_{R_\ell}(o)$. By (10.3), $\lambda_\ell = \lambda_{B_\ell}$ is the first eigenvalue of H on B_ℓ . By Step 1,

$$R < R_1 < \dots < R_\ell < \dots < \hat{R}, \text{ and } \lim_{\ell \rightarrow \infty} R_\ell = \hat{R}.$$

Since $\forall \ell$, $\lambda < \lambda_\ell$, there is a unique $u_\ell \in C(\overline{B_\ell})$, $u_\ell > 0$, radial and non-increasing such that

$$(10.4) \quad H(Du_\ell, D^2u_\ell) + \lambda u_\ell^k = 0, \text{ in } B_\ell, \text{ and } u_\ell(R_\ell) = \delta_\ell,$$

where $\delta_\ell > 0$ is so chosen that $u_\ell(o) = 1$, see (10.2). Since $\lambda_m \geq \lambda_\ell$ and $R_m \leq R_\ell$, for $m \leq \ell$, using Lemma 5.3 in B_m , we see that

$$1 = \frac{u_m(o)}{u_\ell(o)} \leq \frac{u_m(R_m)}{u_\ell(R_m)} \text{ and } 1 = \frac{u_\ell(o)}{u_m(o)} \leq \frac{u_\ell(R_m)}{u_m(R_m)}.$$

Hence, $u_m(R_m) = u_\ell(R_m) = \delta_m$ and $u_m = u_\ell$ in B_m . Thus, u_ℓ extends u_m to B_ℓ , in particular, u_ℓ extends u_0 to B_ℓ . Moreover, δ_ℓ is decreasing.

Step 3: We claim that $\lim_{\ell \rightarrow \infty} \delta_\ell = 0$. Suppose not. Since δ_ℓ is decreasing, $\forall \ell$, $\delta_\ell \geq \eta$, for some $\eta > 0$. Clearly, $u_\ell \geq \eta$. Take $s = 1/2$ in the estimate in Theorem 7.3(i) to see that there is a solution v to

$$H(Dv, D^2v) + \tilde{\lambda}v^k = 0, \text{ in } B_\ell \text{ and } v(R_\ell) = \delta_\ell, \text{ where } \tilde{\lambda} = \lambda + \varepsilon \text{ and } 0 < \varepsilon \leq \frac{\lambda k \eta}{2(1 - \theta \eta/2)}.$$

Here, we may choose $\lambda(1 + k\eta/2) < \tilde{\lambda} < \lambda_\ell$. Since η is independent of ℓ , letting $\ell \rightarrow \infty$ and using Step 2, we obtain a contradiction.

Step 4: Recalling Step 2, for $x \in B_{\hat{R}}(o)$, define $u(x) = \lim_{\ell \rightarrow \infty} u_\ell(x)$. By (10.4), $u \in C(B_{\hat{R}}(o))$ and $H(Du, D^2u) + \lambda u^k = 0$, in $B_{\hat{R}}(o)$. Since u is radial and decreasing, define $u(\hat{R}) = 0$. We have $u \in C(\overline{B_{\hat{R}}})$, since $u(R_\ell) = u_\ell(R_\ell) = \delta_\ell \rightarrow 0$, see Step 3.

Step 5: We scale u as follows. Set $w(\rho) = u(r)$ where $\rho = rR/\hat{R}$. Thus, $w \in C(\overline{B})$, $w > 0$, solves $H(Dw, D^2w) + \lambda(\hat{R}/R)^\gamma w^k = 0$, in $B_R(o)$, and $w(R) = 0$. By Step 1, $\lambda_R = \lambda(\hat{R}/R)^\gamma$ and, thus, w is a first eigenfunction on $B_R(o)$. \square

REFERENCES

- [1] G. Aronson, M. Crandall and P. Juutinen, *A tour of the theory of absolute minimizing functions*, Bull. Amer. Math. Soc., 41 (2004), 439-505.
- [2] M. Bardi and F. Da Lio, *On the strong maximum principle for fully nonlinear degenerate elliptic equations*, Arch. Math. 73 (1999), 276-285.
- [3] M. Belloni and B. Kawohl, *The pseudo-p-Laplacian eigenvalue problem and viscosity solutions as $p \rightarrow \infty$* , ESAIM, Control, Optimisation and Calculus of Variations, January 2004, vol 10, 28-52, DOI: 10.1051/cocv 2003035.
- [4] H. Berestycki, L. Nirenberg and S.R.S Varadhan, *The principal eigenvalue and the maximum principle for second order elliptic operators in general domains*, Comm. Pure. Appl. Math. 47(1) (1994) 47-92.
- [5] T. Bhattacharya, *An elementary proof of the Harnack inequality for non-negative infinity super-harmonic functions*, vol 2001(2001) no 44, 1-8.
- [6] T. Bhattacharya and L. Marazzi, *An eigenvalue problem for the infinity-Laplacian*, Electron Journal of Differential Equations, vol 2013(2013), no 47, 1-30.
- [7] T. Bhattacharya and A. Mohammed, *Inhomogeneous Dirichlet problems involving the infinity-Laplacian*, Advances in Differential Equations, vol 17, nos 3-4, (2012) 225-266.
- [8] I. Birindelli and F. Demengel, *First eigenvalue and maximum principle for fully nonlinear singular operators*, Adv. Differential Equations 11(1) (2006), 91-119.
- [9] M. G. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27(1992) 1-67.
- [10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998 Edition.
- [11] P. Juutinen, *Principal eigenvalues of a very badly degenerate operator and applications*, J. Differential Equations 236 (2007), no. 2, 532-550.
- [12] P. Juutinen, P. Lindqvist and J. J. Manfredi, *The ∞ -eigenvalue problem*, Arch. Rat. Mech. Anal. 148(2) (1999) 89-105.

- [13] P. Juutinen and P. Lindqvist, *On the higher eigenvalues for the ∞ -eigenvalue problem*, Calc. Var. Partial Differential Equations, 23(2005), no 2, 169-192.
- [14] A. Quaas and B. Sirakov, *Principal eigenvalues and Dirichlet problem for fully nonlinear elliptic operators*, Adv. Math 218(2008), no 1, 105-135.

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